BIG DE RHAM-WITT COHOMOLOGY: BASIC RESULTS

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ABSTRACT. Let X be a smooth projective R-scheme, and let R be an étale $\mathbb{Z}$-algebra. As constructed by Hesselholt, we have the absolute big de Rham-Witt complex $\mathbb{W}\Omega^*_X$ of X at our disposal. There is also a relative version $\mathbb{W}\Omega^*_{X/\mathbb{Z}}$ that is characterized by the vanishing of the positive degree part in the case $X = \mathrm{Spec}(\mathbb{Z})$. In this paper we study the hypercohomology of the relative (big) de Rham-Witt complex of X. We show that it is a projective module over the ring of (big) Witt vectors of R, provided that the de Rham cohomology is torsion-free. In addition, we establish a Poincaré duality theorem. Our results rely on an explicit description of the relative de Rham-Witt complex of a smooth λ-ring, which may be of independent interest.

INTRODUCTION

Let X be a scheme over a perfect field k of characteristic $p > 0$. The de Rham-Witt complex $W\Omega^*_{X/k}$ was defined by Illusie [Ill79] relying on ideas of Lubkin, Bloch and Deligne. It is a projective system of complexes of $W(k)$-modules on X, which is indexed by the positive integers. If X is smooth then the hypercohomology of $W_n\Omega^*_{X/k}$ admits a natural comparison isomorphism to the crystalline cohomology of X with respect to $W_n(k)$.

Langer and Zink have extended Illusie's definition of the de Rham-Witt complex to a relative situation, where X is a scheme over $\mathrm{Spec}(R)$ and R is a $\mathbb{Z}_{(p)}$-algebra [LZ04]. If p is nilpotent in R and X is smooth, then they construct a functorial comparison isomorphism

$$H^*(X, W_n\Omega^*_{X/R}) \cong H^*_{crys}(X/W_n(R)).$$

The big de Rham-Witt complex $\mathbb{W}\Omega^*_X$ was introduced by Hesselholt and Madsen [HM01]. The original construction relied on the adjoint functor theorem and has been replaced by a direct and explicit method due to Hesselholt [Hes]. The big de Rham-Witt complex makes sense for arbitrary schemes hence we may consider schemes over more general bases than $\mathrm{Spec}(\mathbb{Z}_{(p)})$. Again, it is a projective system of graded sheaves $[S \mapsto \mathbb{W}_S\Omega^*_X]$ on X, but the index set consists of finite truncation sets; that is, finite subsets S of $\mathbb{N}_{>0}$ having the property that whenever $n \in S$, all (positive) divisors of n are also contained in S.

For the ring of integers, $\mathbb{W}\Omega^*_{\mathbb{Z}}$ has been computed by Hesselholt [Hes]. It vanishes in degree ≥ 2, but $\mathbb{W}\Omega^1_{\mathbb{Z}}$ is non-zero. In this paper we will consider the relative version $\mathbb{W}\Omega^*_{X/\mathbb{Z}}$ of the de Rham-Witt complex, which is constructed from $\mathbb{W}\Omega^*_X$ by killing the ideal generated by $\mathbb{W}\Omega^1_{\mathbb{Z}}$. The relation with the de Rham-Witt complex

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of Langer-Zink is given in Proposition 2.3.4:

$$\mathbb{W}_{\{1,p,\dots,p^{n-1}\}}\Omega^*_{X/\mathbb{Z}} \otimes \mathbb{Z}_{(p)} = W_n\Omega^*_{X\otimes\mathbb{Z}_{(p)}/\mathbb{Z}_{(p)}}.$$

Our purpose is to show that the de Rham-Witt cohomology

$$H^*_{dRW}(X/R)_S \stackrel{\mathrm{def}}{=} H^*(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$$

is as well-behaved as the usual de Rham cohomology. The main theorem of the paper is the following.

Theorem 1 (cf. Theorem 3.3.1)**.** *Let R be an étale $\mathbb{Z}$-algebra. Let X be a smooth and proper R-scheme. Suppose that the de Rham cohomology $H^*_{dR}(X/R)$ of X is torsion-free. Then $H^*(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$ is a finitely generated projective $\mathbb{W}_S(R)$-module for all finite truncation sets S.*

The flatness of $H^*(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$ is not evident. For example, the Witt vector cohomology $H^*(X, \mathbb{W}_S\mathcal{O}_X)$, for $\dim X > 1$, is not a flat $\mathbb{W}_S(R)$-module in general.

In order to prove Theorem 1, we will construct for all maximal ideals $\mathfrak{m}$ of R and $n, j > 0$, a natural (quasi) isomorphism:

$$R\Gamma(\mathbb{W}_{\{1,p,\dots,p^{n-1}\}}\Omega^*_{X/\mathbb{Z}}) \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/\mathfrak{m}^j) \xrightarrow{\cong} R\Gamma(W_n\Omega^*_{X\otimes R/\mathfrak{m}^j/(R/\mathfrak{m}^j)}),$$

where $p = \mathrm{char}(R/\mathfrak{m})$. The right hand side is $R\Gamma$ of the de Rham-Witt complex defined by Langer and Zink. Thus it computes the crystalline cohomology, which in our case is a free $W_n(R/\mathfrak{m}^j)$-module. Taking the limit $\varprojlim_j$, this will yield the flatness of

$$H^*(X, \mathbb{W}_{\{1,p,\dots,p^{n-1}\}}\Omega^*_{X/\mathbb{Z}}) \otimes_{W_n(R)} W_n(\varprojlim_j R/\mathfrak{m}^j)$$

as $W_n(\varprojlim_j R/\mathfrak{m}^j)$-module, which is sufficient in order to prove the flatness of the de Rham-Witt cohomology.

Concerning Poincaré duality we will show the following theorem.

Theorem 2 (cf. Corollary 4.3.3)**.** *Let R be an étale $\mathbb{Z}$-algebra. Let X be a smooth projective R-scheme such that $H^*_{dR}(X/R)$ is torsion-free. Suppose that X is connected, and set $d := \dim X - 1$. If the canonical map*

$$H^i_{dR}(X/R) \to \mathrm{Hom}_R(H^{2d-i}_{dR}(X/R), R)$$

is an isomorphism, then the same holds for the de Rham-Witt cohomology:

$$H^i_{dRW}(X/R)_S \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{W}_S(R)}(H^{2d-i}_{dRW}(X/R)_S, \mathbb{W}_S(R))$$

for all finite truncation sets S.

In fact, de Rham-Witt cohomology is equipped with a richer structure than the $\mathbb{W}(R)$-module structure, coming from the Frobenius morphisms

$$\phi_n : H^*_{dRW}(X/R)_S \to H^*_{dRW}(X/R)_{S/n},$$

for all positive integers n, and where $S/n := \{s \in S \mid ns \in S\}$. These are Frobenius linear maps satisfying $\phi_n \circ \phi_m = \phi_{nm}$.

The relationship with the Frobenius action on the crystalline cohomology of the fibers is as follows. Let $\mathfrak{m}$ be a maximal ideal of R, set $k = R/\mathfrak{m}$ and $p = \mathrm{char}(k)$. If $H^*_{dR}(X/R)$ is torsion-free then there is a natural isomorphism

$$H^i_{dRW}(X/R)_{\{1,p,\dots,p^{n-1}\}} \otimes_{W_n(R)} W_n(k) \cong H^i_{crys}(X \otimes_R k/W_n(k)),$$

and $\phi_p \otimes F_p$ corresponds via this isomorphism to the composition of $H^i_{crys}(\mathrm{Frob})$ with the projection.

As will be made precise in Section 4, the projective system $H^i_{dRW}(X/R)$, together with the Frobenius morphisms $\{\phi_n\}_{n\in\mathbb{N}_{>0}}$, defines an object in a rigid $\otimes$-category $\mathcal{C}_R$. Maybe the most important property of $\mathcal{C}_R$ is the existence of a conservative, faithful $\otimes$-functor

$$T : \mathcal{C}_R \to (R\text{-modules}), \qquad T(H^i_{dRW}(X/R)) = H^i_{dR}(X/R).$$

Moreover, $\mathcal{C}_R$ has Tate objects $\mathbf{1}(m)$, $m \in \mathbb{Z}$, and the first step towards Poincaré duality will be to prove the existence of a natural morphism in $\mathcal{C}_R$:

$$H^{2d}_{dRW}(X/R) \to \mathbf{1}(-d) \qquad (d = \dim X - 1).$$

Then it will follow easily that

$$H^i_{dRW}(X/R) \xrightarrow{\cong} \underline{\mathrm{Hom}}(H^{2d-i}_{dRW}(X/R), \mathbf{1}(-d)),$$

provided that the assumptions of Theorem 2 are satisfied. Taking the underlying $\mathbb{W}(R)$-modules one obtains Theorem 2.

All our results are based on an explicit description of the de Rham-Witt complex $\mathbb{W}_S\Omega^*_{A/\mathbb{Z}}$ for a λ-ring A that is a smooth $\mathbb{Z}$-algebra. Our main example is the polynomial ring $A = \mathbb{Z}[x_1, \ldots, x_d]$. For $i \geq 0$ and $n \in \mathbb{N}_{>0}$, define

$$\Omega^i_{A/\mathbb{Z}}(n) \overset{\mathrm{def}}{=} n \cdot \Omega^i_{A/\mathbb{Z}} + d\Omega^{i-1}_{A/\mathbb{Z}} \subset \Omega^i_{A/\mathbb{Z}}.$$

We will show that

$$\mathbb{W}_S\Omega^i_{A/\mathbb{Z}} \cong \prod_{n\in S} \Omega^i_{A/\mathbb{Z}}(n).$$

Moreover, the differential $d : \mathbb{W}_S\Omega^i_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^{i+1}_{A/\mathbb{Z}}$ corresponds to the product of the maps $n^{-1}d : \Omega^i_{A/\mathbb{Z}}(n) \to \Omega^{i+1}_{A/\mathbb{Z}}(n)$. Finally, the Frobenius morphisms may be described explicitly in terms of the Adams operations of A.

Acknowledgements. After this manuscript had appeared on arXiv, we received a letter from professor J. Borger who informed us that he had already obtained Theorem 1 in collaboration with M. Kisin by using similar methods.

Contents

1. Witt vectors

In this section we give a brief review of Witt vectors where we follow [Hes, §1]. Moreover, we prove some simple corollaries of the Theorem of Borger and van der Kallen (Theorem 1.0.14) that will be used in the following sections.

1.0.1. A subset $S \subset \mathbb{N} = \{1, 2, \dots\}$ is called a *truncation set* if $n \in S$ implies that all positive divisors of n are contained in S. Let A be a commutative ring. For all truncation sets S we have the ring of Witt vectors $\mathbb{W}_S(A)$ at our disposal. As a set we have $\mathbb{W}_S(A) = \prod_{n\in S} A$ (we set $\mathbb{W}_\emptyset(A) = \{0\}$), and the ring structure on $\mathbb{W}_S(A)$ is such that the following two properties hold:

- The ghost map
$$gh = (gh_n)_{n\in S} : \mathbb{W}_S(A) \to \prod_{n\in S} A$$
$$gh_n((a_s)_{s\in S}) := \sum_{d|n} d \cdot a_d^{n/d}$$
is a ring homomorphism, where the target of gh has the product ring structure.
- For a ring homomorphism $\rho : B \to A$, the map
$$\prod_{s\in S} B = \mathbb{W}_S(B) \to \mathbb{W}_S(A) = \prod_{s\in S} A$$
$$(b_s)_{s\in S} \mapsto (\rho(b_s))_{s\in S}$$
is a ring homomorphism. We denote this morphism by $\mathbb{W}_S(\rho)$ or simply ρ.

1.0.2. The ring $\mathbb{W}_S(A)$ is commutative. For us the most important case will be when A is flat over $\mathbb{Z}$. Then the ghost map is injective and we can think of $\mathbb{W}_S(A)$ as a subring of $\prod_{n\in S} A$.

For an inclusion $T \subset S$ of truncation sets, the map $\pi_T : \mathbb{W}_S(A) \to \mathbb{W}_T(A)$ given by
$$\pi_T((a_s)_{s\in S}) := (a_t)_{t\in T}$$
is a ring homomorphism.

1.0.3. For a truncation set S and $n \in \mathbb{N}$ we denote by S/n the truncation set defined by
$$S/n := \{s \in S \mid ns \in S\}.$$
There is a functorial morphism of rings
$$F_n : \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A),$$
called the *Frobenius*, such that
$$gh_s \circ F_n = gh_{sn}, \qquad \text{for all } s \in S/n.$$
For positive integers n, m we have $(S/n)/m = S/nm = (S/m)/n$ and
$$F_m \circ F_n = F_{nm} = F_n \circ F_m.$$

For an inclusion $T \subset S$ of truncation sets, the following diagram is commutative:

(1.0.1)
$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[r, "F_n"] \arrow[d, "\pi_T"'] & \mathbb{W}_{S/n}(A) \arrow[d, "\pi_{T/n}"] \\ \mathbb{W}_T(A) \arrow[r, "F_n"] & \mathbb{W}_{T/n}(A). \end{tikzcd}$$

1.0.4. For a truncation set S and $n \in \mathbb{N}$, the map

$$V_n : \mathbb{W}_{S/n}(A) \to \mathbb{W}_S(A),$$
$$(V_n((a_t)_{t \in S/n}))_s := \begin{cases} 0 & \text{if } n \nmid s, \\ a_{s/n} & \text{if } n \mid s, \end{cases}$$

is a morphism of $\mathbb{W}_S(A)$-modules, where the source is a $\mathbb{W}_S(A)$-module via F_n. This means that

$$V_n(F_n(a) \cdot b) = a \cdot V_n(b)$$

for all $a \in \mathbb{W}_S(A)$ and $b \in \mathbb{W}_{S/n}(A)$.

Furthermore, V_n satisfies

$$gh_s \circ V_n = \begin{cases} n \cdot gh_{s/n} & \text{if } n \mid s, \\ 0 & \text{otherwise}, \end{cases}$$

for all $s \in S$. Again, for positive integers n, m, the equality $V_n \circ V_m = V_{nm} = V_m \circ V_n$ holds. For an inclusion of truncation sets $T \subset S$ the diagram analogous to (1.0.1) is commutative.

1.0.5. For all $n \in \mathbb{N}$ we have

$$F_n \circ V_n = n,$$

that is, $F_n \circ V_n$ equals multiplication with n. Note that $V_n \circ F_n$ is multiplication with $V_n(1)$, and $V_n(1) \neq n$ in general (e.g. $A = \mathbb{Z}$), so that F_n and V_n don't commute.

However, if n, m are positive integers such that $(m, n) = 1$, then

$$F_n \circ V_m = V_m \circ F_n \qquad ((m, n) = 1)$$

considered as morphisms $\mathbb{W}_{S/m}(A) \to \mathbb{W}_{S/n}(A)$.

1.0.6. *Teichmüller map.* For all non-empty truncation sets S we define the map

$$[-] : A \to \mathbb{W}_S(A), \quad a \mapsto [a] := (a, 0, 0, \dots) \in \prod_{s \in S} A = \mathbb{W}_S(A).$$

The element $[a]$ is called the Teichmüller representative of a. The following properties hold:

- For all $a, b \in A$: $[ab] = [a] \cdot [b]$. Moreover, $[1] = 1$ and $[0] = 0$.
- For all $a \in A$ and $n \in \mathbb{N}$: $F_n([a]) = [a^n]$.

If S is a finite truncation set then every element $a \in \mathbb{W}_S(A)$ can be uniquely written as

$$a = \sum_{s \in S} V_s([a_s]) = (a_s)_{s \in S}.$$

Proposition 1.0.7. *Let A be a ring, and let $T \subset A$ be a multiplicative set. Let S be a finite truncation set. We can consider T via the Teichmüller map as multiplicative set in $\mathbb{W}_S(A)$. The natural ring homomorphism*

$$T^{-1}\mathbb{W}_S(A) \to \mathbb{W}_S(T^{-1}A).$$

is an isomorphism.

Proof. We can reduce to the case $T = \{1, t, t^2, \dots\}$. Let us prove the injectivity first. Suppose that $a = \sum_{s\in S} V_s([a_s]) \in \mathbb{W}_S(A)$ is such that its image in $\mathbb{W}_S(T^{-1}A)$ vanishes. This means that the image of a_s in $T^{-1}A$ vanishes for all $s \in S$. Fix $n \geq 1$ with $t^n a_s = 0$ for all $s \in S$. We compute in $\mathbb{W}_S(A)$:

$$\begin{aligned} [t^n] \cdot \sum_{s\in S} V_s([a_s]) &= \sum_{s\in S} V_s(F_s([t^n]) \cdot [a_s]) \\ &= \sum_{s\in S} V_s([t^{ns}][a_s]) \\ &= \sum_{s\in S} V_s([t^{ns}a_s]) = 0. \end{aligned}$$

This proves the injectivity. For the surjectivity we can write every $a \in \mathbb{W}_S(T^{-1}A)$ as $a = \sum_{s\in S} V_s([\frac{a_s}{t^{ns}}])$ with $a_s \in A$ and some n. Therefore $a = [t]^{-n} \cdot \sum_{s\in S} V_s([a_s])$, and is contained in the image of $T^{-1}\mathbb{W}_S(A)$. $\square$

1.0.8. Let S be a truncation set, and let n be a positive integer; set $T := S\backslash\{s \in S; n \mid s\}$. Then T is a truncation set. We have a short exact sequence of $\mathbb{W}_S(A)$-modules:

$$0 \to \mathbb{W}_{S/n}(A) \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{\pi_T} \mathbb{W}_T(A) \to 0. \tag{1.0.2}$$

Lemma 1.0.9. *Let $T \subset \mathbb{Z}$ be a multiplicative set. Let A be a ring, and let S be a finite truncation set. Then*

$$\mathbb{W}_S(A) \otimes_{\mathbb{Z}} T^{-1}\mathbb{Z} \to \mathbb{W}_S(T^{-1}A)$$

is an isomorphism.

Proof. As a first step we need to show that every $t \in T$ is invertible in $\mathbb{W}_S(T^{-1}A)$. For this we certainly may assume that $A = \mathbb{Z}$. Thus we have to prove that

$$(t^{-1}, t^{-1}, t^{-1}, \dots) \in gh(\mathbb{W}_S(T^{-1}\mathbb{Z})).$$

This follows from the Dwork Lemma [Hes, Lemma 1.1] with ϕ_p being the identity for all primes p.

Now that we have a well-defined map, we can use the short exact sequence (1.0.2) to prove the proposition by induction on the length of S. $\square$

1.0.10. We denote by $\mathbb{W}(A)$ the ring $\mathbb{W}_{\mathbb{N}}(A)$. There is a unique ring homomorphism

$$\Delta : \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A)),$$

such that $gh_n \circ \Delta = F_n$ for all positive integers n [Hes, Proposition 1.18].

Definition 1.0.11. [Hes, Definition 1.20] A λ-ring is a pair (A, λ_A) of a ring and a ring homomorphism $\lambda_A : A \to \mathbb{W}(A)$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\lambda_A"] \arrow[dr, "id"'] & \mathbb{W}(A) \arrow[d, "\pi_{\{1\}}"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\lambda_A"] \arrow[d, "\lambda_A"'] & & \mathbb{W}(A) \arrow[d, "\Delta"] \\ \mathbb{W}(A) \arrow[rr, "\mathbb{W}(\lambda_A)"] & & \mathbb{W}(\mathbb{W}(A)). \end{tikzcd}$$

We denote the ring endomorphism $gh_n \circ \lambda_A$ of A by ψ_n, and call it the nth associated Adams operation.

Lemma 1.0.12. *Let A be a λ-ring. The following holds.*

(i) *For all positive integers m, n: $\psi_m \circ \psi_n = \psi_{mn}$. Moreover, $\psi_1 = id_A$.*
(ii) *For all primes p and all $a \in A$: $\psi_p(a) = a^p \mod pA$, i.e. ψ_p is a lifting of the absolute Frobenius.*

Proof. For (i). We apply gh_n to the second diagram in Definition 1.0.11 to obtain $F_n \circ \lambda_A = \lambda_A \circ \psi_n$. Now,

$$gh_m \circ F_n \circ \lambda_A = gh_{mn} \circ \lambda_A = \psi_{mn}$$
$$gh_m \circ \lambda_A \circ \psi_n = \psi_m \circ \psi_n.$$

The equality $\psi_1 = id_A$ follows immediately from the first diagram in the definition and $gh_1 = \pi_{\{1\}}$.

For (ii). Modulo p, we have

$$\psi_p(a) = gh_p(\lambda_A(a)) = gh_1(\lambda_A(a))^p = \psi_1(a)^p = a^p.$$

□

Examples for λ-rings include $\mathbb{Z}$ and $\mathbb{Z}[x_1, \dots, x_d]$. For $\mathbb{Z}$ there is a unique λ-structure. For $A = \mathbb{Z}[x_1, \dots, x_d]$ the ring homomorphism given by $\lambda_A(x_i) = [x_i]$ makes A into a λ-ring.

1.0.13. The following theorem will be very useful throughout the paper.

Theorem 1.0.14. *(Borger-van der Kallen) Let S be a finite truncation set, and let n be a positive integer. Let $\rho : A \to B$ be an étale ring homomorphism. The following holds.*

(1) *The induced ring homomorphism $\mathbb{W}_S(A) \to \mathbb{W}_S(B)$ is étale.*
(2) *The morphism*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A), F_n} \mathbb{W}_{S/n}(A) \to \mathbb{W}_{S/n}(B), \quad b \otimes a \mapsto F_n(b) \cdot \mathbb{W}_{S/n}(\rho)(a),$$

is an isomorphism.

The references for this theorem are [Bor11a, Theorem B] [Bor11b, Corollary 15.4] and [vdK86, Theorem 2.4] (cf. [Hes, Theorem 1.22]).

Lemma 1.0.15. *Let $\rho : A \to B$ is an étale ring homomorphism. For two finite truncation sets $T \subset S$ the map*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_T(A) \to \mathbb{W}_T(B)$$

is an isomorphism.

Proof. We may suppose that $S = T \cup \{n\}$. The exact sequence

$$0 \to A \xrightarrow{V_n} \mathbb{W}_S(A) \to \mathbb{W}_T(A) \to 0$$

yields after applying $\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)}$ (which is an exact functor by Theorem 1.0.14) a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A),F_n} A \arrow[r] \arrow[d] & \mathbb{W}_S(B) \arrow[r] \arrow[d] & \mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_T(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "V_n"] & \mathbb{W}_S(B) \arrow[r] & \mathbb{W}_T(B) \arrow[r] & 0
\end{tikzcd}$$

Again by Theorem 1.0.14, the left vertical map is an isomorphism. □

Corollary 1.0.16. *Let A be an R-algebra, and let $R \to R'$ be an étale ring homomorphism. Let S be a finite truncation set. The natural ring homomorphism*

$$\mathbb{W}_S(A) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R') \to \mathbb{W}_S(A \otimes_R R')$$

is an isomorphism.

Proof. Let $n \in S$ be such that $S/n = \{1\}$. Since $\mathbb{W}_S(R')$ is an étale $\mathbb{W}_S(R)$-algebra, we obtain a morphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
A \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R') \arrow[r] \arrow[d, "V_n \otimes id"] & A \otimes_R R' \arrow[d, "V_n"] \\
\mathbb{W}_S(A) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R') \arrow[r] \arrow[d] & \mathbb{W}_S(A \otimes_R R') \arrow[d] \\
\mathbb{W}_{S\setminus\{n\}}(A) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R') \arrow[r] \arrow[d] & \mathbb{W}_{S\setminus\{n\}}(A \otimes_R R') \arrow[d] \\
0 & 0
\end{tikzcd}$$

By Lemma 1.0.15, we have

$$\mathbb{W}_{S\setminus\{n\}}(A) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R') = \mathbb{W}_{S\setminus\{n\}}(A) \otimes_{\mathbb{W}_{S\setminus\{n\}}(R)} \mathbb{W}_{S\setminus\{n\}}(R').$$

In view of Theorem 1.0.14, the map $A \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R') \to A \otimes_R R'$, which is given by $a \otimes b \mapsto a \otimes F_n(b)$, is an isomorphism. Thus the claim follows by induction. □

We will use the following convention in the next lemma. If a prime p has been fixed then we will write W_n for $\mathbb{W}_{\{1,p,p^2,\dots,p^{n-1}\}}$.

Lemma 1.0.17. *Let p be a prime. Let $A \to B$ be an étale morphism of $\mathbb{Z}_{(p)}$-algebras. For all positive integers n, m, and all non-negative integers s, the map*

$$W_{n+s}(B) \otimes_{W_{n+s}(A),F_{p^s}} W_n(A/p^m) \to W_n(B/p^m), \quad b \otimes a \mapsto F_p^s(b)a$$

is an isomorphism.

Proof. Since

$$\begin{aligned} W_{n+s}(B) \otimes_{W_{n+s}(A),F_{p^s}} W_n(A/p^m) &\to W_n(B) \otimes_{W_n(A)} W_n(A/p^m) \\ b \otimes a &\mapsto F_p^s(b) \otimes a \end{aligned}$$

is an isomorphism, we may assume that $s = 0$. For $n = 1$ the statement is obvious. For $n > 1$ and any ring R, we have an exact sequence

$$0 \to R \xrightarrow{V_p^{n-1}} W_n(R) \to W_{n-1}(R) \to 0. \tag{1.0.3}$$

We have a morphism of short exact sequences

$$W_n(B) \otimes_{W_n(A)} (1.0.3)(R = A/p^m) \to (1.0.3)(R = B/p^m). \tag{1.0.4}$$

Again $W_n(B) \otimes_{W_n(A),F_p^{n-1}} A/p^m \xrightarrow{\cong} B \otimes_A A/p^m = B/p^m$, and in view of Lemma 1.0.15 we have

$$W_n(B) \otimes_{W_n(A)} W_{n-1}(A/p^m) \cong W_{n-1}(B) \otimes_{W_{n-1}(A)} W_{n-1}(A/p^m) \cong W_{n-1}(B/p^m),$$

by induction. Thus (1.0.4) is an isomorphism. □

1.0.18. Let p be a prime. Let R be a $\mathbb{Z}_{(p)}$-algebra. Since all primes different from p are invertible in R, the same holds in $\mathbb{W}_S(R)$ (Lemma 1.0.9). The category of $\mathbb{W}_S(R)$-modules, for a finite truncation set S, factors in the following way. Set

$$\epsilon_{1,S} := \prod_{\substack{\text{primes } \ell \neq p \\ S/\ell \neq \emptyset}} (1 - \frac{1}{\ell} V_\ell(1)) \in \mathbb{W}_S(R),$$

$$\epsilon_{n,S} := \frac{1}{n} V_n\left(\epsilon_{1,S/n}\right) \quad \text{for all } n \geq 1 \text{ with } (n,p) = 1.$$

Of course, if $S/n = \emptyset$ then $\epsilon_{S,n} = 0$. In the following we will simply write ϵ_n for $\epsilon_{n,S}$.

Lemma 1.0.19. (1) *For all positive integers $n \neq n'$ with $(n,p) = 1 = (n',p)$ the equalities*

$$\epsilon_n^2 = \epsilon_n, \qquad \epsilon_n \epsilon_{n'} = 0,$$

hold.

(2) *The equality*

$$\sum_{(n,p)=1, n \geq 1} \epsilon_n = 1$$

holds.

(3) *For all $m, n \geq 1$ with $(m,p) = 1 = (n,p)$ we have*

$$F_m(\epsilon_n) = \begin{cases} \epsilon_{n/m} & \text{if } m \mid n, \\ 0 & \text{if } m \nmid n. \end{cases}$$

Proof. Straightforward. □

Therefore we obtain a decomposition

$$\mathbb{W}_S(R) = \prod_{n \geq 1, (n,p)=1} \epsilon_n \mathbb{W}_S(R). \tag{1.0.5}$$

Notation 1.0.20. For a finite truncation set S we denote by S_p the elements in S that are p-powers, that is $S_p = S \cap \{p^i \mid i \geq 0\}$.

Proposition 1.0.21. *Let R be a $\mathbb{Z}_{(p)}$-algebra. Let S be a finite truncation set.*

(1) *The projection $\pi_{S_p} : \mathbb{W}_S(R) \to \mathbb{W}_{S_p}(R)$ induces an isomorphism*
$$\epsilon_1 \mathbb{W}_S(R) \xrightarrow{\cong} \mathbb{W}_{S_p}(R).$$

(2) *The morphism*
$$\epsilon_n \mathbb{W}_S(R) \to \mathbb{W}_{S/n}(R), \quad \epsilon_n x \mapsto F_n(\epsilon_n x),$$
induces an isomorphism $\epsilon_n \mathbb{W}_S(R) \cong \epsilon_1 \mathbb{W}_{S/n}(R)$ for all n with $(n,p) = 1$.

Proof. For (1). We have $\pi_{S_p}(\epsilon_n) = 0$ for all $n > 1$, since
$$\ker(\pi_{S_p}) = \sum_{n>1,(n,p)=1} V_n(\mathbb{W}_{S/n}(R)).$$
Therefore $\epsilon_1 \mathbb{W}_S(R) \to \mathbb{W}_{S_p}(R)$ is surjective. For all $n > 1$ with $(n,p) = 1$ we have
$$\epsilon_1 \cdot V_n(x) = V_n(F_n(\epsilon_1) \cdot x) = 0$$
by Lemma 1.0.19(3), which implies $\ker(\pi_{S_p}) \cap \epsilon_1 \mathbb{W}_S(R) = 0$.

For (2). Since $F_n(\epsilon_n) = \epsilon_1$, we get a map
$$F_n : \epsilon_n \mathbb{W}_S(R) \to \epsilon_1 \mathbb{W}_{S/n}(R).$$
The inverse is simply given by $\epsilon_1 y \mapsto \epsilon_n \frac{1}{n} V_n(\epsilon_1 y)$. □

Corollary 1.0.22. *Let R be a $\mathbb{Z}_{(p)}$-algebra. For a finite truncation set S, the functor*
$$M \mapsto \bigoplus_{n\geq 1,(n,p)=1} \epsilon_n M$$
defines an equivalence of categories
$$(\mathbb{W}_S(R)\text{-}modules) \xrightarrow{\cong} \prod_{n\geq 1,(n,p)=1} (\mathbb{W}_{(S/n)_p}(R)\text{-}modules).$$

1.0.23. Let R be a $\mathbb{Q}$-algebra. Let S be a finite truncation set. We define
$$\tau_{1,S} := \prod_{\text{primes } \ell} (1 - \frac{1}{\ell} V_\ell(1)) \in \mathbb{W}_S(R),$$
$$\tau_{n,S} := \frac{1}{n} V_n \left(\tau_{1,S/n}\right) \quad \text{for all } n \geq 1.$$

The analogous statements of Lemma 1.0.19 hold. We obtain a decomposition
$$\mathbb{W}_S(R) = \prod_{n\in S} \tau_n \mathbb{W}_S(R), \tag{1.0.6}$$
and
$$\tau_n \mathbb{W}_S(R) \xrightarrow{F_n} \tau_1 \mathbb{W}_{S/n}(R) \xrightarrow{\pi_{\{1\}}} \mathbb{W}_{\{1\}}(R) = R$$
is an isomorphism. The decomposition (1.0.6) is nothing else than the isomorphism
$$\mathbb{W}_S(R) \xrightarrow{(gh_s)_{s\in S}} \prod_{n\in S} R$$
given by the ghost map.

Corollary 1.0.24. *Let R be a $\mathbb{Q}$-algebra. The functor*

$$M \mapsto \bigoplus_{n\geq 1} \tau_n M$$

defines an equivalence of categories

$$(\mathbb{W}_S(R)\text{-modules}) \xrightarrow{\cong} \prod_{n\in S} (R\text{-modules}).$$

1.0.25. For a truncation set S we have

$$\mathbb{W}_S(\mathbb{Z}) = \prod_{n\in S} \mathbb{Z}\cdot V_n(1),$$

and the product is given by $V_m(1)\cdot V_n(1) = c\cdot V_{mn/c}(1)$, where $c=(m,n)$ is the greatest common divisor [Hes, Proposition 1.6].

The following two Lemmas are concerned with maximal ideals in $\mathbb{W}_S(R)$ for an étale $\mathbb{Z}$-algebra R.

Lemma 1.0.26. *Let R be an étale $\mathbb{Z}$-algebra. Let S be a finite truncation set. For every maximal ideal $\mathfrak{m}\subset \mathbb{W}_S(R)$ there exists a maximal ideal $\mathfrak{p}\subset R$ such that $\mathbb{W}_S(R)\to\mathbb{W}_S(R)/\mathfrak{m}$ factors through $\mathbb{W}_S(R_\mathfrak{p})$.*

Proof. Set $k=\mathbb{W}_S(R)/\mathfrak{m}$, we distinguish two cases:

(1) k has characteristic 0,
(2) k has characteristic $p>0$.

The first case can not happen. Indeed, since $\mathbb{W}_S(R)$ is étale over $\mathbb{W}_S(\mathbb{Z})$, it is a finitely generated $\mathbb{W}_S(\mathbb{Z})$-algebra. Because $\mathbb{W}_S(\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-algebra, every closed point of $\mathrm{Spec}(\mathbb{W}_S(R))$ has a finite residue field.

Suppose now that k has characteristic $p>0$. We have a factorization

$$\mathbb{W}_S(R)\to\mathbb{W}_S(R)\otimes_\mathbb{Z}\mathbb{Z}_{(p)}\xrightarrow{=}\mathbb{W}_S(R\otimes_\mathbb{Z}\mathbb{Z}_{(p)})\to k.$$

By decomposing

$$\mathbb{W}_S(R\otimes\mathbb{Z}_{(p)})\xrightarrow{=}\prod_{n\geq 1,(n,p)=1}\epsilon_n\mathbb{W}_S(R\otimes\mathbb{Z}_{(p)})$$
$$\xrightarrow{\cong,\prod_n \pi_{(S/n)_p}\circ F_n}\prod_{n\geq 1,(n,p)=1}\mathbb{W}_{(S/n)_p}(R\otimes\mathbb{Z}_{(p)}),$$

we can reduce to the case where S consists only of p-powers. Since $R\otimes\mathbb{Z}_{(p)}$ is an étale $\mathbb{Z}_{(p)}$-algebra we know that

$$\ker(\pi_{\{1\}}) = \sum_{i>0} V_p^i(1)\cdot\mathbb{W}_S(R\otimes\mathbb{Z}_{(p)}).$$

Finally, $V_{p^i}(1)^2 = p^iV_{p^i}(1)$, hence $V_{p^i}(1)$ maps to zero in k (if $i>0$). Thus $\mathbb{W}_S(R\otimes\mathbb{Z}_{(p)})\to k$ factors through $\mathbb{W}_S(R\otimes\mathbb{Z}_{(p)})\to\mathbb{W}_{\{1\}}(R\otimes\mathbb{Z}_{(p)}) = R\otimes\mathbb{Z}_{(p)}\xrightarrow{\rho} k$. In this case we can take

$$\mathfrak{p} = \ker(R\to R\otimes\mathbb{Z}_{(p)}\xrightarrow{\rho} k).$$

$\square$

Lemma 1.0.27. *Let p be a prime. Let R be an étale $\mathbb{Z}_{(p)}$-algebra. Suppose that R is integral and $pR \neq R$. Let S be a p-typical finite truncation set, i.e. S consists only of p-powers. Then every maximal ideal $\mathfrak{a}$ of $\mathbb{W}_S(R)$ is the form*

$$\mathfrak{a} = \ker(\mathbb{W}_S(R) \xrightarrow{\pi_{\{1\}}} R \to R/\mathfrak{m})$$

for a unique maximal ideal $\mathfrak{m}$ of R.

Proof. Let $\mathfrak{a} \subset \mathbb{W}_S(R)$ be a maximal ideal and set $k = \mathbb{W}_S(R)/\mathfrak{a}$.

If $\mathrm{char}(k) = 0$ then the map $\mathbb{W}_S(R) \to k$ has to factor through a ghost component

$$\mathbb{W}_S(R) \xrightarrow{gh_{p^i}} R \to k$$

for some i. But R doesn't have closed points with residue field of characteristic 0, and we get a contradiction.

Therefore $\mathrm{char}(k) = p$. Since R is an étale $\mathbb{Z}_{(p)}$-algebra we know that

$$\ker(\pi_{\{1\}}) = \sum_{i>0} V_p^i(1) \cdot \mathbb{W}_S(R).$$

And the equality $V_{p^i}(1)^2 = p^i V_{p^i}(1)$ implies that $\mathbb{W}_S(R) \to k$ has to factor through $\pi_{\{1\}}$. □

2. The relative big de Rham-Witt complex for smooth λ-rings

2.1. Construction of an explicit version.

2.1.1. Let A be a smooth $\mathbb{Z}$-algebra, and suppose that A is a λ-ring. Denote by ψ_n, for $n \geq 1$, the Adams operations introduced in Section 1.0.10.

2.1.2. The induced map $\psi_n : \Omega^q_{A/\mathbb{Z}} \to \Omega^q_{A/\mathbb{Z}}$ is divisible by n^q, and $F_n = n^{-q} \cdot \psi_n$ is well-defined on $\Omega^q_{A/\mathbb{Z}}$. The following equalities hold:

- $F_1 = id$,
- $F_n \circ F_m = F_{n \cdot m}$ for all $n, m \geq 1$,
- $d \circ F_n = n \cdot F_n \circ d$ for all $n \geq 1$,
- $d \circ F_n \circ d = 0$ for all $n \geq 1$.

Definition 2.1.3. For $n \geq 1$, we define the complex $\Omega^*_{A/\mathbb{Z}}(n)$ by

$$\Omega^q_{A/\mathbb{Z}}(n) := n \cdot \Omega^q_{A/\mathbb{Z}} + d\Omega^{q-1}_{A/\mathbb{Z}},$$

and differential $\frac{1}{n}d$.

By definition, $\Omega^q_{A/\mathbb{Z}}(n) \subset \Omega^q_{A/\mathbb{Z}}$, and $d(\Omega^q_{A/\mathbb{Z}}(n)) \subset n \cdot \Omega^{q+1}_{A/\mathbb{Z}}$. Since $\Omega^{q+1}_{A/\mathbb{Z}}$ is torsion-free, the differential $\frac{1}{n}d$ is well-defined. Via

$$a \cdot \omega = \psi_n(a)\omega,$$

we obtain an A-module structure on $\Omega^q_{A/\mathbb{Z}}(n)$.

2.1.4. Let us define Frobenius and Verschiebung morphisms. For all positive integers k, we set

$$V_k : \Omega^*_{A/\mathbb{Z}}(n) \to \Omega^*_{A/\mathbb{Z}}(nk), \quad V_k(\omega) = k\omega, \tag{2.1.1}$$

$$F_k : \Omega^*_{A/\mathbb{Z}}(n) \to \Omega^*_{A/\mathbb{Z}}(n/\mathrm{gcd}(n,k)), \quad F_k(\omega) = F_{k/\mathrm{gcd}(n,k)}(\omega). \tag{2.1.2}$$

The Frobenius is well-defined: if $c = \mathrm{gcd}(n,k)$ and i, j are such that $ik + jn = c$, then

$$F_{k/c}(d\omega) = \frac{n}{c} \cdot jF_{k/c}(d\omega) + i\frac{k}{c}F_{k/c}(d\omega) = \frac{n}{c} \cdot jF_{k/c}(d\omega) + d(iF_{k/c}(\omega)) \in \Omega^*_{A/\mathbb{Z}}(n/c)$$

for all $\omega \in \Omega^*_{A/\mathbb{Z}}$.

Lemma 2.1.5. *Let $q \geq 0$ be an integer. Let A be a smooth λ-ring over $\mathbb{Z}$. If $d\eta \in n\Omega^{q+1}_{A/\mathbb{Z}}$ then*

$$\eta = \sum_{e|n} \frac{n}{e} F_e(\eta_e) + \sum_{e|n} F_e(d(\delta_e)),$$

for some η_e, δ_e.

Proof. The case $n = 1$ is obvious. Let $n = p$ be a prime. The Cartier isomorphism reads

$$F_p : \Omega^q_{A/pA} \xrightarrow{\cong} H^q(\Omega^*_{A/pA}) \qquad \text{for all } q \geq 0. \tag{2.1.3}$$

Since $\Omega^q_{A/pA} = \Omega^q_{A/\mathbb{Z}} \otimes_A A/pA$, the case $n = p$ follows from the surjectivity of (2.1.3). Now let $n = p^i$ with $i \geq 2$. By induction we know that

$$\eta = \sum_{j=0}^{i-1} p^{i-1-j} F_{p^j}(\eta_{p^j}) + \sum_{j=0}^{i-1} F_{p^j}(d\delta_{p^j}).$$

Without loss of generality we may assume that $\eta = \sum_{j=0}^{i-1} p^{i-1-j} F_{p^j}(\eta_{p^j})$. Now, $d\eta \in p^i\Omega^{q+1}_{A/\mathbb{Z}}$ implies $p^{i-1}\sum_{j=0}^{i-1} F_{p^j}(d\eta_{p^j}) \in p^i\Omega^{q+1}_{A/\mathbb{Z}}$, hence

$$\sum_{j=0}^{i-1} F_{p^j}(d\eta_{p^j}) \in p\Omega^{q+1}_{A/\mathbb{Z}}. \tag{2.1.4}$$

We claim that (2.1.4) implies $\eta_{p^j} \in p\Omega^q_A + d\Omega^{q-1}_A + F_p(\Omega^q_A)$ for all j, which proves the case $n = p^i$. We can write

$$\sum_{j=0}^{i-1} F_{p^j}(d\eta_{p^j}) = d\eta_1 + F_p(\sum_{j=0}^{i-2} F_{p^j}(d\eta_{p^{j+1}})) \in p\Omega^{q+1}_{A/\mathbb{Z}}.$$

In view of the injectivity of (2.1.3) this implies

$$\sum_{j=0}^{i-2} F_{p^j}(d\eta_{p^{j+1}}) \in p\Omega^{q+1}_{A/\mathbb{Z}}. \tag{2.1.5}$$

Therefore it suffices to show $\eta_1 \in p\Omega^q_A + d\Omega^{q-1}_A + F_p(\Omega^q_A)$. Now, (2.1.5) implies $d\eta_1 \in p\Omega^{q+1}_A$ and the claim follows from the case $n = p$.

Suppose that $n = n_1 n_2$ with $(n_1, n_2) = 1$ and $n > n_1, n_2$. By induction we know that

$$\eta = \sum_{e|n_1} \frac{n_1}{e} F_e(\eta_e) + \sum_{e|n_1} F_e(d(\delta_e)),$$
$$\eta = \sum_{e|n_2} \frac{n_2}{e} F_e(\eta_e) + \sum_{e|n_2} F_e(d(\delta_e)).$$

Write $1 = an_1 + bn_2$ and

$$\eta = \sum_{e|n_2} \frac{n}{e} F_e(a\eta_e) + \sum_{e|n_2} F_e(d(an_1\delta_e)) + \sum_{e|n_1} \frac{n}{e} F_e(b\eta_e) + \sum_{e|n_1} F_e(d(bn_2\delta_e)).$$

This proves the claim. □

Definition 2.1.6. Let A be a smooth λ-ring over $\mathbb{Z}$. We set

$$\tilde{W}\Omega^q_{A/\mathbb{Z}} := \prod_{n\geq 1} \Omega^q_{A/\mathbb{Z}}(n).$$

For a truncation set S we set

$$\tilde{W}_S\Omega^q_{A/\mathbb{Z}} := \prod_{n\in S} \Omega^q_{A/\mathbb{Z}}(n). \tag{2.1.6}$$

We will consider $\tilde{W}_S\Omega^q_{A/\mathbb{Z}}$ as a quotient of $\tilde{W}\Omega^q_{A/\mathbb{Z}}$ for the obvious map. We want to show that the system

$$S \mapsto \tilde{W}_S\Omega^*_{A/\mathbb{Z}}$$

is a Witt complex in the sense of [Hes, §4]. For this we want to equip $\tilde{W}\Omega^*_{A/\mathbb{Z}}$ with a structure of a differential graded algebra. Here the degree q part is given by $\tilde{W}\Omega^q_{A/\mathbb{Z}}$. The differential

$$d : \tilde{W}\Omega^q_{A/\mathbb{Z}} \to \tilde{W}\Omega^{q+1}_{A/\mathbb{Z}}$$

is defined by $d = (\frac{d}{n})_{n\geq 1}$ (as in Definition 2.1.3).

For $n, m \geq 1$ let c be the greatest common divisor, write $in + jm = c$. We define

$$\begin{aligned} \Omega^{q_1}_{A/\mathbb{Z}}(n) \times \Omega^{q_1}_{A/\mathbb{Z}}(m) &\to \Omega^{q_1+q_2}_{A/\mathbb{Z}}(nm/c) \\ (\omega_1, \omega_2) &\mapsto F_{m/c}(\omega_1) \cdot F_{n/c}(\omega_2). \end{aligned} \tag{2.1.7}$$

For this to be well-defined we need to show that $F_{m/c}(\omega_1) \cdot F_{n/c}(\omega_2)$ is contained in $\Omega^{q_1+q_2}_{A/\mathbb{Z}}(nm/c)$:

$$\begin{aligned} F_{m/c}(na_1) \cdot F_{n/c}(ma_2 + db_2) &= n \cdot F_{m/c}(a_1)F_{n/c}(db_2) && \text{mod } nm/c \\ &= (in/c + jm/c)n \cdot F_{m/c}(a_1)F_{n/c}(db_2) && \text{mod } nm/c \\ &= in^2/c \cdot F_{m/c}(a_1)F_{n/c}(db_2) && \text{mod } nm/c \\ &= in \cdot F_{m/c}(a_1)d(F_{n/c}(b_2)) && \text{mod } nm/c \\ &= (-1)^{\deg(a_1)} d\left(in \cdot F_{m/c}(a_1)F_{n/c}(b_2)\right) \\ &\quad - (-1)^{\deg(a_1)} in \cdot d(F_{m/c}(a_1)) \cdot F_{n/c}(b_2) && \text{mod } nm/c \\ &= d\left((-1)^{\deg(a_1)} in \cdot F_{m/c}(a_1)F_{n/c}(b_2)\right) && \text{mod } nm/c, \end{aligned}$$

$$\begin{aligned} F_{m/c}(db_1)\cdot F_{n/c}(db_2) &= (in/c+jm/c)\cdot F_{m/c}(db_1)\cdot F_{n/c}(db_2) \\ &= i\cdot F_{m/c}(db_1)\cdot d(F_{n/c}(b_2)) + j\cdot d(F_{m/c}(b_1))\cdot F_{n/c}(db_2) \\ &= d\left((-1)^{\deg(b_1)+1} iF_{m/c}(db_1)F_{n/c}(b_2) + jF_{m/c}(b_1)F_{n/c}(db_2)\right). \end{aligned}$$

It is easy to see that (2.1.7) makes $\tilde{W}\Omega^*_{A/\mathbb{Z}}$ into a differential graded algebra. For every truncation set S, the kernel of the projection $\tilde{W}\Omega^*_{A/\mathbb{Z}} \to \tilde{W}_S\Omega^*_{A/\mathbb{Z}}$ is a two-sided ideal, hence $S \mapsto \tilde{W}_S\Omega^*_{A/\mathbb{Z}}$ defines a contravariant functor from truncation sets to differential graded algebras.

2.1.7. We want to show that there is a natural isomorphism $\eta : \mathbb{W}(A) \to \tilde{W}\Omega^0_{A/\mathbb{Z}}$, inducing an isomorphism

$$\eta_S : \mathbb{W}_S(A) \xrightarrow{\cong} \tilde{W}_S\Omega^0_{A/\mathbb{Z}} \tag{2.1.8}$$

for every truncation set S.

Since A is a λ-ring there is a natural ring homomorphism $\rho : A \to \mathbb{W}(A)$ splitting the projection $\mathbb{W}(A) \to A$ and such that $F_n \circ \rho = \rho \circ \psi_n$ for all $n \geq 1$. Define

$$\mu : \tilde{W}\Omega^0_{A/\mathbb{Z}} \to \mathbb{W}(A), \quad \mu((na_n)_{n\geq 1}) := \sum_{n\geq 1} V_n(\rho(a_n)).$$

Lemma 2.1.8. *The map μ is an isomorphism of rings and is compatible with the F_n, V_n-action for all $n \geq 1$.*

Proof. Let c be the greatest common divisor of n, m. We have

$$V_n(\rho(a_n))\cdot V_m(\rho(a_m)) = V_{nm/c}(\rho(c\psi_{m/c}(a_n)\psi_{n/c}(a_m))),$$

hence μ is a ring homomorphism. Both rings, $\tilde{W}\Omega^0_{A/\mathbb{Z}}$ and $\mathbb{W}(A)$, are complete with respect to the descending filtrations

$$\begin{aligned} \left(\tilde{W}\Omega^0_{A/\mathbb{Z}}\right)^i &:= \prod_{n\geq i} nA, \\ \mathbb{W}(A)^i &= \sum_{n\geq i} V_n(\mathbb{W}(A)). \end{aligned}$$

Since μ is continuous, it is sufficient to check that the graded pieces are isomorphic, which is obvious because ρ is a section of the projection $\mathbb{W}(A) \to A$. Compatibility with the F_n and V_n-action are obvious. $\square$

We define η as the inverse of μ. It induces an isomorphism η_S for every truncation set S, that is simply given by

$$\eta_S(\sum_{n\in S} V_n(\rho(a_n))) = (na_n)_{n\in S}.$$

2.1.9. Let S be a truncation set. For all $n \geq 1$, we have

$$F_n : \tilde{W}\Omega^q_{A/\mathbb{Z}} \to \tilde{W}\Omega^q_{A/\mathbb{Z}}, \quad V_n : \tilde{W}\Omega^q_{A/\mathbb{Z}} \to \tilde{W}\Omega^q_{A/\mathbb{Z}}$$

from Section 2.1.4. Consider $\tilde{W}_S\Omega^q_{A/\mathbb{Z}}$ and $\tilde{W}_{S/n}\Omega^q_{A/\mathbb{Z}}$ as quotients of $\tilde{W}\Omega^q_{A/\mathbb{Z}}$. Then F_n and V_n induce

$$F_n : \tilde{W}_S\Omega^q_{A/\mathbb{Z}} \to \tilde{W}_{S/n}\Omega^q_{A/\mathbb{Z}}, \quad V_n : \tilde{W}_{S/n}\Omega^q_{A/\mathbb{Z}} \to \tilde{W}_S\Omega^q_{A/\mathbb{Z}}. \tag{2.1.9}$$

It is easy to verify that for all positive integers n, m the following identities hold:

$$\begin{aligned} &F_1 = V_1 = id, && F_m F_n = F_{nm}, && V_n V_m = V_{nm}, \\ &F_n V_n = n, && F_m V_n = V_n F_m \quad \text{if } (n,m) = 1, && \\ &F_n \eta_S = \eta_{S/n} F_n, && \eta_S V_n = V_n \eta_{S/n}. && \end{aligned}$$

Lemma 2.1.10. *Let n be a positive integer. Let S be a truncation set.*

(i) *If $x \in \tilde{W}_S \Omega^q_{A/\mathbb{Z}}$ and $y \in \tilde{W}_{S/n} \Omega^{q'}_{A/\mathbb{Z}}$ then*

$$x \cdot V_n(y) = V_n(F_n(x) y).$$

(ii) *If $y \in \tilde{W}_{S/n} \Omega^q_{A/\mathbb{Z}}$ then*

$$F_n d V_n(y) = d(y).$$

Proof. The identities (i) and (ii) are straightforward. □

2.1.11. Recall that we have the ghost map

$$gh : \mathbb{W}(A) \to \prod_{n=1}^{\infty} A.$$
$$gh(a_1, a_2, \dots) = (\sum_{e|n} e a_e^{n/e})_{n \geq 1}.$$

Since A is torsion-free the ghost map is injective. The composition $gh \circ \eta^{-1}$: $\tilde{W}\Omega^0_{A/\mathbb{Z}} \to \prod_{n=1}^{\infty} A$ is simply given by

$$(gh \circ \eta^{-1})((n a_n)_{n \geq 1}) = (\sum_{e|n} e \psi_{n/e}(a_e))_{n \geq 1},$$

or equivalently,

$$(gh \circ \eta^{-1})((a_n)_{n \geq 1}) = (\sum_{e|n} \psi_{n/e}(a_e))_{n \geq 1}.$$

It will be convenient to have a ghost map for the entire complex $\tilde{W}\Omega^*_{A/\mathbb{Z}}$. We won't be able to get a compatibility with Frobenius and Verschiebung, but we can capture the multiplication and the differential. For $q \geq 0$ we define

$$gh : \tilde{W}\Omega^q_{A/\mathbb{Z}} \to \prod_{n=1}^{\infty} \Omega^q_{A/\mathbb{Z}} \tag{2.1.10}$$
$$gh((\omega_n)_n) = (n^q \cdot \sum_{e|n} F_{n/e}(\omega_e))_n.$$

Via the product structure, $\prod_{n=1}^{\infty} \Omega^*_{A/\mathbb{Z}}$ is a differential graded algebra.

Proposition 2.1.12. *The ghost map is an injective morphism of differential graded algebras.*

Proof. Injectivity is obvious. Compatibility with the differential:

$$\begin{aligned} d(n^q \cdot \sum_{e|n} F_{n/e}(\omega_e)) &= n^q \cdot \sum_{e|n} d(F_{n/e}(\omega_e)) \\ &= n^q \cdot \sum_{e|n} n/e \cdot F_{n/e}(d\omega_e) \\ &= n^{q+1} \cdot \sum_{e|n} F_{n/e}(\frac{d}{e}\omega_e). \end{aligned}$$

Compatibility with the multiplication is a straightforward computation. □

We define

$$\begin{gathered} F_k : \prod_{n=1}^{\infty} \Omega^q_{A/\mathbb{Z}} \to \prod_{n=1}^{\infty} \Omega^q_{A/\mathbb{Z}}, \\ F_k((\omega_n)_n) = (\omega_{nk})_n. \end{gathered}$$

The ghost map does not commute with the Frobenius, but we have the following compatibility: $F_k \circ gh = k^q \cdot gh \circ F_k$.

Lemma 2.1.13. *Let S be a truncation set. For all positive integers k and $a \in A$,*

$$F_k d\eta_S([a]) = \eta_{S/k}([a]^{k-1}) d\eta_{S/k}([a]).$$

Proof. It is sufficient to consider $S = \mathbb{N}$, that is, we will work with $\tilde{W}\Omega^*_{A/\mathbb{Z}}$, because the claim follows from this case after projecting. We need to prove

$$F_k d\eta([a]) = \eta([a]^{k-1}) d\eta([a]).$$

In view of Proposition 2.1.12 the claim is implied by

$$k \cdot gh(F_k d\eta([a])) = k \cdot gh(\eta([a]^{k-1}) d\eta([a])).$$

For the left hand side we compute:

$$\begin{aligned} k \cdot gh(F_k d\eta([a])) &= F_k d(gh([a])) \\ &= F_k d((a^n)_n) \\ &= F_k((na^{n-1}da)_n) \\ &= (nka^{nk-1}da)_n. \end{aligned}$$

On the other hand:

$$\begin{aligned} k \cdot gh(\eta([a]^{k-1}) d\eta([a])) &= (ka^{(k-1)n})_n \cdot d((a^n)_n) \\ &= (ka^{(k-1)n})_n \cdot (na^{n-1}da)_n \\ &= (kna^{kn-1}da)_n. \end{aligned}$$

□

Corollary 2.1.14. *The system of differential graded algebras*

$$S \mapsto \tilde{W}_S\Omega^*_{A/\mathbb{Z}}$$

together with η_S (2.1.8) and the Frobenius and Verschiebung (2.1.9) form a Witt complex in the sense of [Hes, §4].

Since the big de Rham-Witt complex is the initial Witt complex, Corollary 2.1.14 implies the existence of a unique morphism of Witt complexes

$$[S \mapsto \mathbb{W}_S\Omega^*_A] \to [S \mapsto \tilde{W}_S\Omega^*_{A/\mathbb{Z}}]. \tag{2.1.11}$$

Lemma 2.1.15. *The morphism (2.1.11) is an epimorphism.*

Proof. It is sufficient to prove that the map $\mathbb{W}_S\Omega^*_A \to \tilde{W}_S\Omega^*_{A/\mathbb{Z}}$ is surjective if S is a finite truncation set. We prove the statement by induction on the length of S. Set $m = \max\{n \mid n \in S\}$. Certainly, $S' := S\backslash\{m\}$ is a truncation set and $S/m = \{1\}$. By induction it suffices to prove that $\Omega^q_{A/\mathbb{Z}}(m)$ is contained in the image of $\mathbb{W}_S\Omega^q_A$. Let $m\omega + d\eta \in \Omega^q_{A/\mathbb{Z}}(m)$ and consider $\omega \in \mathbb{W}_{\{1\}}\Omega^q_A$, $\eta \in \mathbb{W}_{\{1\}}\Omega^{q-1}_A$. Then $V_m\omega + dV_m\eta \in \mathbb{W}_S\Omega^*_A$ has $m\omega + d\eta$ as image. $\square$

2.1.16. Let A be a smooth λ-ring. Our goal is to show that the map (2.1.11) induces an isomorphism

$$\mathbb{W}_S\Omega^*_A/\left(\mathbb{W}_S\Omega^1_{\mathbb{Z}} \cdot \mathbb{W}_S\Omega^*_A\right) \cong \tilde{W}_S\Omega^*_{A/\mathbb{Z}}$$

for every finite truncation set S. Note that $\mathbb{W}_S\Omega^i_{\mathbb{Z}} = 0$ for $i > 1$ by [Hes, Theorem 6.1].

As a first step, let us prove that $\mathbb{W}_S\Omega^1_{\mathbb{Z}}$ maps to zero in $\tilde{W}_S\Omega^*_{A/\mathbb{Z}}$. In view of [Hes, Theorem 6.1], $\mathbb{W}_S\Omega^1_{\mathbb{Z}}$ is generated by elements of the form $d(V_n(1))$ with $n \in S$. Since $V_n(1)$ maps to the image of the element $n \in \Omega^0_{A/\mathbb{Z}}(n)$ via the inclusion $\Omega^0_{A/\mathbb{Z}}(n) \to \tilde{W}_S\Omega^0_{A/\mathbb{Z}}$ (see (2.1.6)), the image of $dV_n(1)$ is zero.

2.2. **Relative big de Rham-Witt complexes (over $\mathbb{Z}$).**

Definition 2.2.1. Let A be a (commutative) ring. Let S be a truncation set and $q \geq 0$. We define

$$\mathbb{W}_S\Omega^q_{A/\mathbb{Z}} := \varprojlim_{\substack{T\subset S\\ T \text{ finite}}} \mathbb{W}_T\Omega^q_A/\left(\mathbb{W}_T\Omega^1_{\mathbb{Z}} \cdot \mathbb{W}_T\Omega^{q-1}_A\right).$$

As usual, we write $\mathbb{W}\Omega^q_{A/\mathbb{Z}}$ for $\mathbb{W}_{\mathbb{N}}\Omega^q_{A/\mathbb{Z}}$.

Again, note that $\mathbb{W}_S\Omega^i_{\mathbb{Z}} = 0$ for $i > 1$ by [Hes, Theorem 6.1]. By definition, we get an induced isomorphism

$$\eta_S : \mathbb{W}_S(A) \to \mathbb{W}_S\Omega^0_{A/\mathbb{Z}}.$$

Proposition 2.2.2. *Let S be a truncation set. Let n be a positive integer. Induced by $\mathbb{W}\Omega^*_A$, we get maps*

$$d : \mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^{q+1}_{A/\mathbb{Z}} \tag{2.2.1}$$

$$F_n : \mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_{S/n}\Omega^q_{A/\mathbb{Z}} \tag{2.2.2}$$

$$V_n : \mathbb{W}_{S/n}\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{A/\mathbb{Z}}. \tag{2.2.3}$$

Moreover, $S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}$ forms a Witt complex.

Proof. For (2.2.1): follows immediately from $d(x \cdot y) = d(x) \cdot y + (-1)^{\deg(x)} x \cdot d(y)$.

For (2.2.2): follows immediately from $F_n(x \cdot y) = F_n(x) \cdot y + x \cdot F_n(y)$.

For (2.2.3): for a finite truncation set T, $\mathbb{W}_T\Omega^1_{\mathbb{Z}}$ is generated by elements of the form $d(V_m(1))$. We have

$$V_n(d(V_m(1)) \cdot y) = V_n(F_n dV_{nm}(1) \cdot y) = dV_{nm}(1) \cdot V_n(y).$$

Thus (2.2.3) is well-defined.

All properties that are needed for $S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}$ to be a Witt complex are inherited from the big de Rham-Witt complex. $\square$

Lemma 2.2.3. *Let n be a positive integer, let S be a truncation set, and let q be a non-negative integer. The following holds.*

(i) *As morphisms $\mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^{q+1}_{A/\mathbb{Z}}$: $V_nF_nd = dV_nF_n$.*
(ii) *As morphisms $\mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^{q+2}_{A/\mathbb{Z}}$: $d^2 = 0$.*
(iii) *As morphisms $\mathbb{W}_{S/n}\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^{q+2}_{A/\mathbb{Z}}$: $dV_nd = 0$.*

Proof. For (i). On the left hand side:

$$V_nF_nd(x) = V_n(F_nd(x) \cdot 1) = d(x) \cdot V_n(1).$$

On the right hand side:

$$dV_nF_n(x) = d(x \cdot V_n(1)) = d(x) \cdot V_n(1) + (-1)^{\deg(x)} x \cdot dV_n(1),$$

and $dV_n(1) = 0$ in $\mathbb{W}_S\Omega^1_{A/\mathbb{Z}}$.

For (ii). We have $d^2(x) = d\log[-1] \cdot dx$, and $d\log[-1] = [-1] \cdot d[-1] = 0$.

For (iii). We have

$$dV_nd(x) = dV_n(F_ndV_n(x)) = d(dV_n(x) \cdot V_n(1)) = \\ d^2V_n(x) \cdot V_n(1) + (-1)^{\deg(x)+1} dV_n(x) \cdot dV_n(1) = 0,$$

in view of (ii). $\square$

Proposition 2.2.4. *The Witt complex $S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}$ is the initial object in the category of Witt complexes over A with $\mathbb{W}(\mathbb{Z})$-linear differential.*

Proof. Let $S \mapsto E^*_S$ be a Witt complex over A with $\mathbb{W}(\mathbb{Z})$-linear differential, that is, $d(a\omega) = ad(\omega)$ for $a \in \mathbb{W}_S(\mathbb{Z})$ and $\omega \in E^*_S$. We only need to show that the canonical morphism

$$[S \mapsto \mathbb{W}_S\Omega^*_A] \to [S \mapsto E^*_S]$$

factors through $[S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}]$. It is enough to check this for finite truncation sets. Since $\mathbb{W}_S\Omega^1_{\mathbb{Z}}$ is generated by elements of the form $dV_n(1)$ with $n \in S$, this follows immediately. $\square$

2.2.5. Let us show that for a scheme X and a finite truncation set S, the assignment $\mathrm{Spec}(A) \mapsto \mathbb{W}_S\Omega^q_{A/\mathbb{Z}}$ for all open affine subsets $\mathrm{Spec}(A)$ of X defines a sheaf. We will need the following lemma.

Lemma 2.2.6. *Let S be a finite truncation set. Let $A \to B$ be an étale morphism of rings. For all $q \geq 0$ the induced morphism of $\mathbb{W}_S(B)$-modules*

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{B/\mathbb{Z}}$$

is an isomorphism.

Proof. By definition we have an exact sequence of $\mathbb{W}_S(A)$-modules

$$\mathbb{W}_S\Omega^{q-1}_A \otimes_{\mathbb{W}_S(\mathbb{Z})} \mathbb{W}_S\Omega^1_{\mathbb{Z}} \to \mathbb{W}_S\Omega^q_A \to \mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \to 0 \tag{2.2.4}$$

By [Hes, Theorem C] we have $\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S\Omega^q_A \xrightarrow{\cong} \mathbb{W}_S\Omega^q_B$ for all q, and the claim follows from the exact sequence (2.2.4) after tensoring with $\mathbb{W}_S(B)\otimes_{\mathbb{W}_S(A)}$. □

Proposition 2.2.7. *Let X be a scheme. Let S be a finite truncation set and $q \geq 0$. The assignment $\mathrm{Spec}(A) \mapsto \mathbb{W}_S\Omega^q_{A/\mathbb{Z}}$, for all open $\mathrm{Spec}(A)$ of X, defines a sheaf which we denote by $\mathbb{W}_S\Omega^q_{X/\mathbb{Z}}$.*

Proof. Certainly it suffices to consider an affine scheme $X = \mathrm{Spec}(A)$. We need to show that for every affine covering $\{\mathrm{Spec}(A_i)\}_{i\in I}$ of X the sequence

$$0 \to \mathbb{W}_S\Omega_{A/\mathbb{Z}} \to \prod_i \mathbb{W}_S\Omega_{A_i/\mathbb{Z}} \to \prod_{i<j} \mathbb{W}_S\Omega_{A_i\otimes_A A_j/\mathbb{Z}} \tag{2.2.5}$$

is exact. We can reduce to a finite index set I since X is quasi-compact.

We know that the Čech-complex

$$0 \to \mathbb{W}_S(A) \to \prod_i \mathbb{W}_S(A_i) \to \prod_{i<j} \mathbb{W}_S(A_i \otimes_A A_j) \to \dots \tag{2.2.6}$$

is an acyclic bounded complex. By Theorem 1.0.14, $\mathbb{W}_S(B)$ is étale (hence flat) over $\mathbb{W}_S(A)$ provided that B is an étale A-algebra. Therefore tensoring (2.2.6) with $\otimes_{\mathbb{W}_S(A)}\mathbb{W}_S\Omega^q_{A/\mathbb{Z}}$ is an acyclic complex again. We conclude by using Lemma 2.2.6. □

2.2.8. Let A be a λ-ring and smooth over $\mathbb{Z}$. So far we have a morphism of Witt complexes

$$[S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}] \to [S \mapsto \tilde{W}_S\Omega^*_{A/\mathbb{Z}}].$$

We would like to show that this is an isomorphism. Let us define the inverse. Since Witt complexes are determined by their values on finite truncation sets, it is sufficient to define the inverse only for them.

By Hesselholt's construction of the de Rham-Witt complex [Hes, §3] there is a natural morphism of differential graded algebras

$$\Omega^*_{\mathbb{W}(A)/\mathbb{Z}} \to \mathbb{W}\Omega^*_{A/\mathbb{Z}},$$

because $d\log[-1] = [-1]d[-1] = 0$ in $\mathbb{W}\Omega^1_{A/\mathbb{Z}}$. Recall that we have a ring homomorphism

$$\rho : A \to \mathbb{W}(A),$$

hence an induced map $\rho : \Omega^*_{A/\mathbb{Z}} \to \Omega^*_{\mathbb{W}(A)/\mathbb{Z}} \to \mathbb{W}\Omega^*_{A/\mathbb{Z}}$. We denote by ρ_S the induced map

$$\rho_S : \Omega^*_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}.$$

Lemma 2.2.9. *For all $q \geq 0$ and all positive integers n, the map $\rho : \Omega^q_{A/\mathbb{Z}} \to \mathbb{W}\Omega^q_{A/\mathbb{Z}}$ satisfies $\rho \circ F_n = F_n \circ \rho$.*

Proof. In view of [Hes, Theorem 2.13] we have

$$F_n : \Omega^1_{\mathbb{W}(A)/\mathbb{Z}} \to \Omega^1_{\mathbb{W}(A)/\mathbb{Z}}$$

at our disposal, and $\Omega^1_{\mathbb{W}(A)/\mathbb{Z}} \to \mathbb{W}\Omega^1_{A/\mathbb{Z}}$ is compatible with F_n. Clearly it suffices to treat the cases $q = 0, 1$; the case $q = 0$ holds by assumption.

For the case $q = 1$. In view of the formula for F_n given in [Hes, Theorem 2.13] we have to show

$$F_n(da) = \sum_{e|n} \rho_e(a)^{\frac{n}{e}-1} d(\rho_e(a)) \qquad \text{for all } a \in A,$$

where ρ_e is the e-th component of $\rho : A \to \mathbb{W}(A)$ via the identity $\mathbb{W}(A) = \prod_{n \in \mathbb{N}_{>0}} A$. This is implied by the following equalities:

$$\begin{aligned} nF_n(da) &= d(\psi_n(a)), \\ n \sum_{e|n} \rho_e(a)^{\frac{n}{e}-1} d(\rho_e(a)) &= d(\sum_{e|n} e\rho_e(a)^{\frac{n}{e}}) \\ &= d(gh_n \circ \rho(a)) = d(\psi_n(a)). \end{aligned}$$

□

Lemma 2.2.10. *Let n be a positive integer. Let S be a truncation set such that $n \in S$. The map*

$$\begin{aligned} &\Omega^q_{A/\mathbb{Z}}(n) \to \mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \\ &n\omega + d\eta \mapsto V_n\rho_{S/n}(\omega) + dV_n\rho_{S/n}(\eta), \end{aligned}$$

is well-defined.

Proof. In view of Lemma 2.1.5, the equality $n\omega + d\eta = 0$ implies the existence of η_e, δ_e, for all divisors e of n, such that

$$\eta = \sum_{e|n} \frac{n}{e} F_e(\eta_e) + \sum_{e|n} F_e(d(\delta_e)),$$

hence

$$\omega = -\sum_{e|n} F_e(d\eta_e).$$

Now,

$$\begin{aligned} dV_n\rho_{S/n}(F_e d(\delta_e)) &= dV_n F_e d\rho_{S/(n/e)}(\delta_e) \\ &= dV_{n/e} V_e F_e d\rho_{S/(n/e)}(\delta_e) \\ &= dV_{n/e} dV_e F_e \rho_{S/(n/e)}(\delta_e) \quad \text{(Lemma 2.2.3(i))} \\ &= 0 \quad \text{(Lemma 2.2.3(iii))}. \end{aligned}$$

Moreover,

$$\begin{aligned} dV_n\rho_{S/n}(\frac{n}{e}F_e(\eta_e)) &= dV_n \frac{n}{e} F_e \rho_{S/(n/e)}(\eta_e) \\ &= dV_{n/e} \frac{n}{e} V_e F_e \rho_{S/(n/e)}(\eta_e) \\ &= V_{n/e} dV_e F_e \rho_{S/(n/e)}(\eta_e) \\ &= V_n F_e d\rho_{S/(n/e)}(\eta_e) \quad \text{(Lemma 2.2.3(i))} \\ &= V_n \rho_{S/n}(F_e d\eta_e), \end{aligned}$$

which finishes the proof. □

Lemma 2.2.10 allows us to define

$$\epsilon_S : \tilde{W}_S\Omega^q_{A/\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{A/\mathbb{Z}}$$
$$(n\omega_n + d\eta_n)_{n\in S} \mapsto \sum_{n\in S} V_n\rho(\omega_n) + dV_n\rho(\eta_n),$$

for every finite truncation set S. We define ϵ_S for every truncation set S by taking the limit.

Proposition 2.2.11. *The maps*

$$[S \mapsto \tilde{W}_S\Omega^q_{A/\mathbb{Z}}] \xrightarrow{[S\mapsto\epsilon_S]} [S \mapsto \mathbb{W}_S\Omega^q_{A/\mathbb{Z}}]$$

form a morphism of Witt systems.

Proof. We may work with finite truncation sets S only. By definition of $\eta_S : \mathbb{W}_S(A) \to \tilde{W}_S\Omega^0_{A/\mathbb{Z}}$ (Section 2.1.7) the compatibility $\epsilon_S \circ \eta_S = \eta_S$ is obvious.

For the compatibility with d:

$$d\epsilon_S(n\omega + d\eta) = d(V_n\rho(\omega) + dV_n\rho(\eta)) = dV_n\rho(\omega)$$
$$\epsilon_S d(n\omega + d\eta) = \epsilon_S(d\omega) = dV_n\rho(\omega).$$

For the compatibility with V_n:

$$V_n\epsilon_{S/n}(m\omega + d\eta) = V_n(V_m\rho(\omega) + dV_m\rho(\eta)) = V_{nm}\rho(\omega) + ndV_{nm}\rho(\eta)$$
$$\epsilon_S(V_n(m\omega + d\eta)) = \epsilon_S(nm\omega + nd\eta) = V_{nm}\rho(\omega) + ndV_{nm}\rho(\eta).$$

For the compatibility with F_n recall that F_n is defined by

$$F_n : \Omega^q_A(m) \to \Omega^q_A(m/c)$$
$$m\omega + d\eta \mapsto F_{n/c}(m\omega + d\eta),$$

where $c = gcd(m, n)$. Writing $i\frac{n}{c} + j\frac{m}{c} = 1$, we have

$$F_{n/c}(m\omega + d\eta) = \frac{m}{c}(c \cdot F_{n/c}(\omega) + j \cdot F_{n/c}(d\eta)) + d(i \cdot F_{n/c}(\eta)).$$

Noting that $F_{n/c}$ and $V_{m/c}$ commute, we get

$$\begin{aligned}
\epsilon_{S/n}(\frac{m}{c}&(c \cdot F_{n/c}\omega + j \cdot F_{n/c}d\eta) + d(i \cdot F_{n/c}(\eta))) \\
&= V_{m/c}(c \cdot F_{n/c}\rho(\omega) + j \cdot F_{n/c}d\rho(\eta)) + i \cdot dV_{m/c}F_{n/c}\rho(\eta) \\
&= cV_{m/c}F_{n/c}\rho(\omega) + j\frac{m}{c}F_{n/c}dV_{m/c}\rho(\eta) + i\frac{n}{c}F_{n/c}dV_{m/c}\rho(\eta) \\
&= cV_{m/c}F_{n/c}\rho(\omega) + F_{n/c}dV_{m/c}\rho(\eta).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
F_n\epsilon_S(m\omega + d\eta) &= F_nV_m\rho(\omega) + F_ndV_m\rho(\eta) \\
&= cF_{n/c}V_{m/c}\rho(\omega) + F_{n/c}dV_{m/c}\rho(\eta).
\end{aligned}$$

This implies the compatibility with the Frobenius. □

Theorem 2.2.12. *The morphism of Witt systems*

$$[S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}] \to [S \mapsto \tilde{W}_S\Omega^*_{A/\mathbb{Z}}]$$

is an isomorphism.

Proof. Proposition 2.2.11 implies the existence of a morphism of Witt system in the other direction. For both Witt systems the canonical morphism from $[S \mapsto \mathbb{W}_S\Omega^*_A]$ is an epimorphism (Lemma 2.1.15), hence the claim. $\square$

2.3. **Applications.** In this section we collect applications of Theorem 2.2.12, that will be used in the remainder of the paper.

Proposition 2.3.1. *Let A and B be smooth $\mathbb{Z}$-algebras. Let p be a prime, and let S be a finite truncation set.*

(1) *Via the equivalence from Corollary 1.0.22 for $R = \mathbb{Z}_{(p)}$ we have*

$$\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \mapsto \bigoplus_{n\geq 1,(n,p)=1} \mathbb{W}_{(S/n)_p}\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

(2) *For a morphism $f : A \to B$ the induced morphism $f_S : \mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathbb{W}_S\Omega^*_{B/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ maps to*

$$f_S \mapsto \bigoplus_{n\geq 1,(n,p)=1} f_{(S/n)_p}$$

via the equivalence from Corollary 1.0.22.

Proof. For (1). Note that $\mathbb{W}_S(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{W}_S(\mathbb{Z}_{(p)})$ (Lemma 1.0.9) and therefore the statement makes sense, because $\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a complex of $\mathbb{W}_S(\mathbb{Z}_{(p)})$-modules.

The first step is to show that the projection induces an isomorphism of complexes

$$\epsilon_1(\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \to \mathbb{W}_{S_p}\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \tag{2.3.1}$$

(see Notation 1.0.20 for S_p, and Section 1.0.18 for the definition of ϵ_1). In view of Proposition 2.2.7 we can reduce to the case where there exists an étale morphism $B \to A$ with $B = \mathbb{Z}[x_1, \ldots, x_d]$. The isomorphism from Lemma 2.2.6 implies

$$\epsilon_1(\mathbb{W}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \otimes_{\epsilon_1(\mathbb{W}_S(B)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)})} \epsilon_1(\mathbb{W}_S\Omega^q_{B/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \xrightarrow{\cong} \epsilon_1(\mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$$

for all q. Since $\mathbb{W}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{W}_S(A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ for any ring A (Lemma 1.0.9), Proposition 1.0.21 and Lemma 2.2.6 allow us to reduce to the case $A = B$.

Theorem 2.2.12 implies

$$\mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \prod_{n\in S} \Omega^q_{A/\mathbb{Z}}(n) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

for any finite truncation set S. For $n \in S$ and $\omega \in \Omega^q_{A/\mathbb{Z}}(n)$ it suffices to show that

$$\epsilon_1\omega = 0, \quad \text{if } n \notin S_p$$

$$\epsilon_1\omega - \omega \in \prod_{s\notin S_p} \Omega^q_{A/\mathbb{Z}}(s) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \quad \text{if } n \in S_p.$$

Suppose $n \notin S_p$, and let $\ell \neq p$ be a prime that divides n. Then $V_\ell(1) \cdot \omega = \ell \cdot \omega$, hence $\epsilon_1\omega = 0$.

If $n \in S_p$ then $\epsilon_1\omega$ can only have non-zero components at $\alpha \cdot n$ with $(\alpha, p) = 1$. It is easy to see that the n-th component of $\epsilon_1\omega$ equals ω. Thus we proved that (2.3.1) is an isomorphism.

The isomorphism

$$\epsilon_n(\mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \xrightarrow{n^q F_n} \epsilon_1(\mathbb{W}_{S/n}\Omega^q_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}),$$

for $n \geq 1$ with $(n,p)=1$, follows from $F_n(\epsilon_n)=\epsilon_1$ and the existence of the inverse map $\frac{\epsilon_n}{n^{q+1}}V_n$. We introduced the factor n^q in order to obtain an isomorphism of complexes

$$\epsilon_n(\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \xrightarrow{\cong} \epsilon_1(\mathbb{W}_{S/n}\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}).$$

For (2). Follows immediately from the construction in (1). $\square$

Remark 2.3.2. Proposition 2.3.1 may be also true if A, B are not assumed to be smooth.

The interest in $\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ comes from the following fact.

Proposition 2.3.3. *Let A be a ring, let p be a prime, and set $A' := A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. There is a unique isomorphism*

$$[S \mapsto \mathbb{W}_S\Omega^*_{A'/\mathbb{Z}}] \xrightarrow{\tau} [S \mapsto \varprojlim_{\substack{T \subset S \\ T \text{ finite}}} \mathbb{W}_T\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}]$$

of Witt complexes over A'.

Proof. It is sufficient to consider finite truncation sets. By Lemma 1.0.9 we know that

$$\mathbb{W}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{W}_S(A'), \tag{2.3.2}$$

and all the other properties of a Witt complex over A' are easily proved for $[S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}]$. The differential is $[S \mapsto \mathbb{W}_S(\mathbb{Z})]$-linear, hence the existence of a unique morphism. On the other hand, $[S \mapsto \mathbb{W}_S\Omega^*_{A'/\mathbb{Z}}]$ is a Witt complex over A (with $[S \mapsto \mathbb{W}_S(\mathbb{Z})]$-linear differential). Thus we get a morphism of Witt complexes over A:

$$[S \mapsto \varprojlim_{\substack{T \subset S \\ T \text{ finite}}} \mathbb{W}_T\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}] \xrightarrow{\epsilon} [S \mapsto \mathbb{W}_S\Omega^*_{A'/\mathbb{Z}}].$$

It is automatically a morphism of Witt complexes over A' (in view of (2.3.2)) and thus $\epsilon \circ \tau = id$. Moreover, we know that $\xi = \tau \circ \epsilon \circ \xi$, where

$$\xi : [S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}] \to [S \mapsto \varprojlim_{\substack{T \subset S \\ T \text{ finite}}} \mathbb{W}_T\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}]$$

is the canonical morphism. This implies $\tau \circ \epsilon = id$. $\square$

Proposition 2.3.4. *Let A be a $\mathbb{Z}_{(p)}$-algebra. Then*

$$n \mapsto \mathbb{W}_{\{1,p,\ldots,p^{n-1}\}}\Omega^*_{A/\mathbb{Z}}$$

is the relative de Rham-Witt complex $n \mapsto W_n\Omega^*_{A/\mathbb{Z}_{(p)}}$ defined by Langer and Zink [LZ04].

Proof. We have a functor

$$f : (\text{Witt systems over } A \text{ with } \mathbb{W}(\mathbb{Z})\text{-linear differential}) \to (F\text{-}V\text{-procomplexes over the } \mathbb{Z}_{(p)}\text{-algebra } A),$$

where we use the definition of [Hes, §4] for the source category and the definition of [LZ04, Introduction] for the target category. The functor is defined by

$$[S \mapsto P_S] \mapsto [n \mapsto P_{\{1,p,\ldots,p^{n-1}\}}].$$

This functor admits a right adjoint E defined as follows. Let $[n \mapsto P_n]$ be a F-V-procomplex. We set $E(P)_{\{1,p,\ldots,p^{n-1}\}} := P_n$. For any finite truncation set S we set

$$E(P)_S = \prod_{n\geq 1,(n,p)=1} E(P)_{(S/n)_p}$$

equipped with the product ring structure (see Notation 1.0.20 for $(S/n)_p$). For $T \subset S$ the map $E(P)_S \to E(P)_T$ is defined by $E(P)_{(S/n)_p} \to E(P)_{(T/n)_p}$ induced by $(T/n)_p \subset (S/n)_p$ for all n.

In order to define

$$\eta_S : \mathbb{W}_S(A) \to E(P)^0,$$

we have to use Proposition 1.0.21. Explicitly, we have a ring isomorphism

$$\mathbb{W}_S(A) \to \prod_{n\geq 1,(n,p)=1} \mathbb{W}_{(S/n)_p}(A), \quad \sum_n \epsilon_n a_n \mapsto (\pi_{(S/n)_p} F_n(\epsilon_n a_n))_n,$$

where $\pi_{(S/n)_p} : \mathbb{W}_{S/n}(A) \to \mathbb{W}_{(S/n)_p}(A)$ is the projection. The differential d of $E(P)$ is defined by

$$d(e_n)_n := (\frac{1}{n} d(e_n))_n,$$

where n runs over all positive integers n such that $(n,p) = 1$. We set

$$\begin{aligned} F_p &: E(P)_S \to E(P)_{S/p}, \quad F_p((e_n)_n) = (F_p(e_n))_n, \\ F_\ell &: E(P)_S \to E(P)_{S/\ell}, \quad F_\ell((e_n)_n)_{n'} = e_{n'\ell} \quad \text{for a prime } \ell \neq p. \end{aligned}$$

We set

$$V_p : E(P)_{S/p} \to E(P)_S, \quad V_p((e_n)_n) = (V_p(e_n))_n,$$

$$\text{for a prime } \ell \neq p\text{: } V_\ell : E(P)_{S/\ell} \to E(P)_S, \quad V_\ell((e_n)_n)_{n'} = \begin{cases} \ell \cdot e_{n'/\ell} & \text{if } \ell \mid n' \\ 0 & \text{otherwise.} \end{cases}$$

In general we define $F_n := \prod_i F_{\ell_i}^{\nu_i}$ and $V_n := \prod_i V_{\ell_i}^{\nu_i}$ if $n = \prod_i \ell_i^{\nu_i}$. It's a straightforward calculation to prove that $E(P)$ defines a Witt complex over A with $\mathbb{W}(\mathbb{Z})$-linear differential. We thus obtain a functor

$$\begin{aligned} E : (&F\text{-}V\text{-procomplexes over the } \mathbb{Z}_{(p)}\text{-algebra } A) \to \\ &\qquad (\text{Witt systems over } A \text{ with } \mathbb{W}(\mathbb{Z})\text{-linear differential}) \end{aligned}$$

that is easily seen to be right-adjoint to f. Therefore $f([S \mapsto \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}])$ is the initial object in the category of F-V-procomplexes as is the relative de Rham-Witt complex constructed by Langer and Zink [LZ04]. □

Proposition 2.3.5. *Let A and B be smooth $\mathbb{Z}$-algebras. Let S be a finite truncation set.*

(1) *Via the equivalence from Corollary 1.0.24 for $R = \mathbb{Q}$ we have*

$$\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \mapsto \bigoplus_{n\in S} \Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{2.3.3}$$

(2) *For a morphism $f : A \to B$ the induced morphism $f_S : \mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{W}_S\Omega^*_{B/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ maps to*

$$f_S \mapsto \bigoplus_{n\in S} f_{\{1\}}$$

via the equivalence from Corollary 1.0.24.

Proof. For (1). For every truncation set T we denote by $\pi : \mathbb{W}_T\Omega^*_{A/\mathbb{Z}} \to \Omega^*_{A/\mathbb{Z}}$ the projection. The morphism (2.3.3) is induced by

$$\tau_n(\mathbb{W}_S\Omega^q_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{n^q\pi\circ F_n} \Omega^q_{A/\mathbb{Z}} \otimes \mathbb{Q}.$$

Again the factor n^q has been added in order to obtain a morphism of complexes

$$\tau_n(\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}) \to \Omega^*_{A/\mathbb{Z}} \otimes \mathbb{Q}.$$

By using Lemma 2.2.6 and Proposition 2.2.7 we may suppose that $A = \mathbb{Z}[x_1, \dots, x_d]$. Fix a λ-structure on A. Via the isomorphism of Theorem 2.2.12 the morphism (2.3.3) agrees with the map (2.1.10). Now it is easy to see that (2.3.3) is an isomorphism.

Statement (2) is obvious. □

Proposition 2.3.6. *Let X be a smooth scheme over $\mathbb{Z}$. Let $q \geq 0$ be an integer and let S be a finite truncation set. Then $\mathbb{W}_S\Omega^q_X$ is $\mathbb{Z}$-torsion-free, that is, multiplication by a non-zero integer is injective.*

Proof. It is sufficient to prove that the sequence

$$0 \to \mathbb{W}_S\Omega^q_B \xrightarrow{\cdot n} \mathbb{W}_S\Omega^q_B$$

is exact (where n is a non-zero integer) for every open $\mathrm{Spec}(B)$ of X that admits an étale morphism $\mathrm{Spec}(B) \to \mathbb{A}^d_{\mathbb{Z}} = \mathrm{Spec}(A)$. In view of Lemma 2.2.6 and the fact that $\mathbb{W}_S(B)$ is étale over $\mathbb{W}_S(A)$, we can reduce to the case $B = A$. In this case, Theorem 2.2.12 gives an explicit description of $\mathbb{W}_S\Omega^q_B$ that is obviously torsion-free. □

Proposition 2.3.7. *Let p be a prime. Set $A = \mathbb{Z}[x_1, \dots, x_d]$ and $A' := A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let n be a non-negative integer; set $S = \{1, p, p^2, \dots, p^n\}$. Recall that we have a map of complexes $\Omega^*_{A/\mathbb{Z}} = \Omega^*_{A/\mathbb{Z}}(0) \subset \mathbb{W}_S\Omega^*_{A/\mathbb{Z}}$ coming from the λ-structure on $\mathbb{Z}[x_1, \dots, x_d]$ with Adams operators $\psi_n(x_i) = x_i^n$ for all i. The induced morphism*

$$\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{W}_S(\mathbb{Z}_{(p)}) \to \mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \tag{2.3.4}$$

is a quasi-isomorphism. Equivalently,

$$\Omega^*_{A'/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{W}_S(\mathbb{Z}_{(p)}) \to \mathbb{W}_S\Omega^*_{A'/\mathbb{Z}} \tag{2.3.5}$$

is a quasi-isomorphism.

Proof. Note that

$$\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{W}_S(\mathbb{Z}_{(p)}) = \Omega^*_{A'/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{W}_S(\mathbb{Z}_{(p)}),$$

thus Proposition 2.3.3 implies the equivalence of the statements.

We have a decomposition as a complex:

$$\Omega^*_{A'/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{W}_S(\mathbb{Z}_{(p)}) = \bigoplus_{i=0}^{n} \Omega^*_{A'/\mathbb{Z}_{(p)}} V_{p^i}(1).$$

Recall that by Theorem 2.2.12 we have a decomposition as a complex:

$$\mathbb{W}_S\Omega^*_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \bigoplus_{i=0}^{n} \Omega^*_{A/\mathbb{Z}}(p^i) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

We observe

$$\Omega^*_{A/\mathbb{Z}}(p^i) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = p^i \Omega^*_{A'/\mathbb{Z}_{(p)}} + d\Omega^*_{A'/\mathbb{Z}_{(p)}} \subset \Omega^*_{A'/\mathbb{Z}_{(p)}},$$

with differential $d(p^i\omega + d\eta) = d\omega$. The morphism (2.3.4) respects the decompositions, and is given by

$$\Omega^*_{A'/\mathbb{Z}_{(p)}} V_{p^i}(1) \to p^i \Omega^*_{A'/\mathbb{Z}_{(p)}} + d\Omega^*_{A'/\mathbb{Z}_{(p)}}, \quad \omega V_{p^i}(1) \mapsto p^i F_{p^i}(\omega).$$

Since F_p, acting on $\Omega^*_{A'/\mathbb{Z}_{(p)}}$, is injective, it suffices to show that the complex

$$\left(p^i \Omega^*_{A'/\mathbb{Z}_{(p)}} + d\Omega^*_{A'/\mathbb{Z}_{(p)}}\right) / p^i F_p^i \Omega^*_{A'/\mathbb{Z}_{(p)}}$$

is acyclic. Equivalently, for all $q \geq 0$, we have the inclusion

$$p^i\{\omega \in \Omega^q_{A'/\mathbb{Z}_{(p)}} \mid d\omega \in p^i F_p^i \Omega^{q+1}_{A'/\mathbb{Z}_{(p)}}\} \subset p^i F_p^i \Omega^q_{A'/\mathbb{Z}_{(p)}} + d\Omega^{q-1}_{A'/\mathbb{Z}_{(p)}}. \tag{2.3.6}$$

In order to prove (2.3.6) by induction on i, we will need Lemma 2.3.8 below. For $i = 1$ and $d\omega \in pF_p\Omega^{q+1}$, the Lemma implies immediately that $\omega \in F_p\Omega^q + d\Omega^{q-1}$.

Suppose that $i > 1$ and $d\omega \in p^i F_p^i \Omega^{q+1}$. By induction we conclude the existence of η and ϵ such that

$$p^{i-1}\omega = p^{i-1} F_p^{i-1} \eta + d\epsilon.$$

Since F_p is injective, it is easy to see that $d\eta \in pF_p\Omega^{q+1}$, which implies $\eta = F_p\eta_1 + d\eta_2$ for some η_1, η_2. Thus

$$p^i\omega = p^i F_p^{i-1}(F_p\eta_1 + d\eta_2) + pd\epsilon = p^i F_p^i \eta_1 + pd(F_p^{i-1}(\eta_2) + \epsilon).$$

□

Lemma 2.3.8. *Let $A = \mathbb{Z}_{(p)}[x_1, \ldots, x_d]$. There is a decomposition of complexes*

$$\Omega^*_{A/\mathbb{Z}_{(p)}} = F_p \Omega^*_{A/\mathbb{Z}_{(p)}} \oplus C^*, \tag{2.3.7}$$

such that C^* is an acyclic complex.

Proof. As complex C^* we can take

$$C^q := \sum_I \sum_J \mathbb{Z}_{(p)} x^J dx_{I_1} \wedge \cdots \wedge dx_{I_q}$$

where the sum runs over all indices (I, J) such that there exists a k with

$$J_k \not\equiv \begin{cases} 0 \mod p & \text{if } J_k \notin I \\ -1 \mod p & \text{if } J_k \in I. \end{cases}$$

It is easy to see that the decomposition (2.3.7) holds. In order to see that C^* is acyclic we define

$$\deg(x^J dx_{I_1} \wedge \cdots \wedge dx_{I_q}) = |J| + q.$$

In this way we get a decomposition of complexes

$$\Omega^*_{A/\mathbb{Z}_{(p)}} = \bigoplus_{n \geq 0} (\Omega^*_{A/\mathbb{Z}_{(p)}})_{\deg = n},$$

where $(\Omega^*_{A/\mathbb{Z}_{(p)}})_{\deg=n}$ are the homogeneous forms of degree n. Clearly, for $n \in \mathbb{Z}$ there exists an m (depending on n) such that

$$p^m\omega \in d\Omega^* \qquad \text{if } d\omega = 0 \text{ and } \deg(\omega) = n. \tag{2.3.8}$$

Let us prove that C^* is acyclic. For $\omega \in C^q_{\deg=n}$ with $d\omega = 0$, Cartier's Lemma implies $\omega = d\eta + p\omega_1$. We can take η and ω_1 to be homogeneous of degree n and contained in C^*. Again $d\omega_1 = 0$ and we conclude $\omega_1 = d\eta_1 + p\omega_2$. Finally, we get

$$\omega = d(\eta + \sum_{i=1}^{m-1} p^i \eta_i) + p^m \omega_m,$$

and (2.3.8) implies $\omega \in dC^*$. □

3. Cohomology of the relative big de Rham-Witt complex

3.1. Reduction modulo powers of p.

3.1.1. In order to simplify the notation we will use the following convention. If a prime p has been fixed then we will write W_n for $\mathbb{W}_{\{1,p,\dots,p^{n-1}\}}$. For example, $W_n(A)$ is the ring of p-typical Witt vectors of A of length n. The goal of this section is to prove the following theorem.

Theorem 3.1.2. *Let R be an étale $\mathbb{Z}_{(p)}$-algebra, let B be a smooth R-algebra, and let n, m be positive integers. Choose a $W_n(R)$-free resolution*

$$T := \ldots \to T^{-2} \to T^{-1} \to T^0$$

of $W_n(R/p^m)$. There exists a functorial quasi-isomorphism of complexes of $W_n(R)$-modules

$$W_n\Omega^*_{B/\mathbb{Z}} \otimes_{W_n(R)} T \to W_n\Omega^*_{(B/p^m)/(R/p^m)}, \tag{3.1.1}$$

where the right hand side is the relative de Rham-Witt complex defined by Langer and Zink. In particular, we obtain an isomorphism

$$W_n\Omega^*_{B/\mathbb{Z}} \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/p^m) \xrightarrow{\cong} W_n\Omega^*_{(B/p^m)/(R/p^m)}, \tag{3.1.2}$$

in the derived category of $W_n(R)$-modules.

More precisely, functoriality means that for any morphism $A \to B$ of R-algebras, the diagram

$$\begin{array}{ccc} W_n\Omega^*_{B/\mathbb{Z}} \otimes_{W_n(R)} T & \longrightarrow & W_n\Omega^*_{(B/p^m)/(R/p^m)} \\ \uparrow & & \uparrow \\ W_n\Omega^*_{A/\mathbb{Z}} \otimes_{W_n(R)} T & \longrightarrow & W_n\Omega^*_{(A/p^m)/(R/p^m)} \end{array}$$

is commutative.

Remark 3.1.3. In view of Proposition 2.3.4 we have

$$W_n\Omega^*_{B/\mathbb{Z}} = W_n\Omega^*_{B/\mathbb{Z}_{(p)}},$$

the right hand side being the relative de Rham-Witt complex of Langer and Zink. The proof of Theorem 3.1.2 does not go beyond the methods of [LZ04], so that the theorem may be well-known but we couldn't provide a reference.

3.1.4. The importance of Theorem 3.1.2 for us comes from the comparison isomorphism [LZ04, Theorem 3.1] with crystalline cohomology. If X is a smooth scheme over R/p^m then

$$H^i(X, W_n\Omega^*_{X/(R/p^m)}) \cong H^i((X/W_n(R/p^m))_{crys}, \mathcal{O}_{X/W_n(R/p^m)}).$$

Suppose that R is an étale $\mathbb{Z}_{(p)}$-algebra. Then there exists a canonical ring homomorphism

$$\rho : R \to R/p^{nm} \to W_n(R/p^m).$$

Let X be a smooth scheme over R. The reduction $X_m = X \otimes_R R/p^m$ has a canonical lifting to $W_n(R/p^m)$ defined by $X \otimes_{R,\rho} W_n(R/p^m)$. By the comparison isomorphism of crystalline cohomology with de Rham cohomology due to Berthelot-Ogus we get

$$H^i((X/W_n(R/p^m))_{crys}, \mathcal{O}_{X/W_n(R/p^m)}) \cong H^i_{dR}(X \otimes_{R,\rho} W_n(R/p^m)/W_n(R/p^m)).$$

In particular, if $H^*_{dR}(X/R)$ is a free R-module then

$$H^i((X/W_n(R/p^m))_{crys}, \mathcal{O}_{X/W_n(R/p^m)}) \cong H^i_{dR}(X/R) \otimes_{R,\rho} W_n(R/p^m)$$

is a free $W_n(R/p^m)$-module.

Proof of Theorem 3.1.2. We define the morphism (3.1.1) by

$$W_n\Omega^*_{B/\mathbb{Z}} \otimes_{W_n(R)} T \ \to \ W_n\Omega^*_{B/\mathbb{Z}} \otimes_{W_n(R)} W_n(R/p^m) \ \to \ W_n\Omega^*_{(B/p^m)/(R/p^m)},$$

using Proposition 2.3.4, and the functoriality of the relative de Rham-Witt complex of Langer-Zink. So that the functoriality of (3.1.1) is obvious.

1.Step: The first step is the reduction to $R = \mathbb{Z}_{(p)}$. By [LZ04, Proposition 1.9] we have $W_n\Omega^*_{(B/p^m)/(R/p^m)} = W_n\Omega^*_{(B/p^m)/(\mathbb{Z}/p^m)}$. Lemma 1.0.17 implies

$$W_n(\mathbb{Z}/p^m) \otimes_{W_n(\mathbb{Z}_{(p)})} W_n(R) \xrightarrow{\cong} W_n(R/p^m),$$

and since $W_n(R)$ is étale over $W_n(\mathbb{Z}_{(p)})$, we see that

$$W_n\Omega^*_{B/\mathbb{Z}} \otimes^{\mathbb{L}}_{W_n(\mathbb{Z}_{(p)})} W_n(\mathbb{Z}/p^m) \to W_n\Omega^*_{B/\mathbb{Z}} \otimes^{\mathbb{L}}_{W_n(R)} W_n(R/p^m)$$

is a quasi-isomorphism. Thus we may suppose $R = \mathbb{Z}_{(p)}$.

2.Step: The second step is the reduction to $B = \mathbb{Z}_{(p)}[x_1, \dots, x_d]$. We can use the to Proposition 2.2.7 analogous fact for the relative de Rham-Witt complex of Langer-Zink (which follows from [LZ04, Proposition 1.7]) in order to reduce to the case where there exists an étale morphism $A = \mathbb{Z}_{(p)}[x_1, \dots, x_d] \to B$.

Note that $p^{nm} = 0$ in $W_n(\mathbb{Z}/p^m)$. Since $W_n\Omega^*_{B/\mathbb{Z}}$ is p-torsion-free (Proposition 2.3.6 and 2.3.3), we see that

$$W_n\Omega^*_{B/\mathbb{Z}} \otimes^{\mathbb{L}}_{W_n(\mathbb{Z}_{(p)})} W_n(\mathbb{Z}/p^m) \to W_n\Omega^*_{B/\mathbb{Z}}/p^{nm} \otimes^{\mathbb{L}}_{W_n(\mathbb{Z}_{(p)})/p^{nm}} W_n(\mathbb{Z}/p^m) \tag{3.1.3}$$

is a quasi-isomorphism. Clearly, morphism (3.1.2) factors through (3.1.3). It will be easier to work modulo p^{nm}, because $dF_p^{nm} = p^{nm}F_p^{nm}d$ vanishes modulo p^{nm}.

Set $c = nm + n$, we claim that

$$\begin{aligned} \left(W_c(B)/p^{nm} \otimes_{W_c(A)/p^{nm}} W_n\Omega^*_{A/\mathbb{Z}}/p^{nm}, id \otimes d\right) &\to (W_n\Omega^*_{B/\mathbb{Z}}/p^{nm}, d) \\ b \otimes \omega &\mapsto F_p^{nm}(b) \cdot \omega, \end{aligned} \tag{3.1.4}$$

is an isomorphism of complexes. Note that $W_c(A)$ acts on $W_n\Omega^*_{A/\mathbb{Z}}/p^{nm}$ via $W_c(A) \xrightarrow{F_p^{nm}} W_n(A)$, and therefore (3.1.4) is a morphism of complexes. Theorem 1.0.14 implies that

$$W_c(B) \otimes_{W_c(A)} M \xrightarrow{\cong} W_n(B) \otimes_{W_n(A)} M, \quad b \otimes m \mapsto F_p^{nm}(b) \otimes m,$$

is an isomorphism for all $W_n(A)$-modules M. Thus the claim follows from Lemma 2.2.6.

On the other hand, Lemma 1.0.17 shows that for every $W_n(A/p^m)$-module M the map

$$W_c(B)/p^{nm} \otimes_{W_c(A)/p^{nm}} M \to W_n(B/p^m) \otimes_{W_n(A/p^m)} M, \quad b \otimes m \mapsto F_p^{nm}(b) \otimes m,$$

is an isomorphism. This yields together with [LZ04, Proposition 1.7] an isomorphism of complexes

$$\Big(W_c(B)/p^{nm} \otimes_{W_c(A)/p^{nm}} W_n\Omega^*_{(A/p^m)/(\mathbb{Z}/p^m)}, id \otimes d\Big) \to (W_n\Omega^*_{(B/p^m)/(\mathbb{Z}/p^m)}, d).$$

Finally, since $W_c(B)/p^{nm}$ is étale over $W_c(A)/p^{nm}$, we are reduced to proving that

$$W_n\Omega^*_{A/\mathbb{Z}}/p^{nm} \otimes^{\mathbb{L}}_{W_n(\mathbb{Z}_{(p)})/p^{nm}} W_n(\mathbb{Z}/p^m) \to W_n\Omega^*_{(A/p^m)/(\mathbb{Z}/p^m)}$$

is a quasi-isomorphism.

3.Step: Proof of the case $A = B = \mathbb{Z}_{(p)}[x_1, \ldots, x_d]$. In this case we know that the composition

$$\Omega^*_{A/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W_n(\mathbb{Z}/p^m) \to \Omega^*_{W_n(A/p^m)/W_n(\mathbb{Z}/p^m)} \to W_n\Omega^*_{(A/p^m)/(\mathbb{Z}/p^m)} \tag{3.1.5}$$

is a quasi-isomorphism [LZ04, Theorem 3.5]. Here the first morphism comes from some Frobenius lifting, which induces a ring homomorphism $A \to W(A)$; the second morphism is the unique morphism of differential graded algebras that induces the identity in degree zero and is $W_n(A/p^m)$-linear. We want to fix the first morphism as being induced by the λ-structure $A \to \mathbb{W}(A)$ from Proposition 2.3.7; that is, the Frobenius lifting is simply given by $x_i \mapsto x_i^p$.

Finally, we have a commutative diagram

$$\begin{tikzcd}
\Big(\Omega^*_{A/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W_n(\mathbb{Z}_{(p)})\Big) \otimes^{\mathbb{L}}_{W_n(\mathbb{Z}_{(p)})} W_n(\mathbb{Z}/p^m) \arrow[r, "\text{quis}"] \arrow[d, "\text{quis}"] & \Omega^*_{A/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W_n(\mathbb{Z}/p^m) \arrow[d, "(3.1.5)"] \\
W_n\Omega^*_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_{W_n(\mathbb{Z}_{(p)})} W_n(\mathbb{Z}/p^m) \arrow[r, "(3.1.2)"] & W_n\Omega^*_{(A/p^m)/(\mathbb{Z}/p^m)}.
\end{tikzcd}$$

The left vertical arrow is a quasi-isomorphism because of Proposition 2.3.7. The upper horizontal arrow is a quasi-isomorphism because $\Omega^*_{A/\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W_n(\mathbb{Z}_{(p)})$ is a complex of flat $W_n(\mathbb{Z}_{(p)})$-modules. □

3.2. **Finiteness.**

3.2.1. The goal of this section is to prove the following theorem.

Theorem 3.2.2. *Let R be an étale $\mathbb{Z}$-algebra. Let X be a smooth and proper scheme over R. The following holds.*

(i) *For every finite truncation set S and non-negative integers i, j the cohomology group $H^i(X, \mathbb{W}_S\Omega^j_{X/\mathbb{Z}})$ is a finitely generated $\mathbb{W}_S(R)$-module.*

(ii) *For all $i \geq \dim X$ and $j \geq 0$, we have $H^i(X, \mathbb{W}_S\Omega^j_{X/\mathbb{Z}}) = 0$.*

Via the Hodge-to-de-Rham spectral sequence we obtain the following corollary.

Corollary 3.2.3. *With the assumptions of Theorem 3.2.2, the cohomology groups $H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$ are finitely generated $\mathbb{W}_S(R)$-modules.*

Our proof of Theorem 3.2.2 is a brute-force induction on the length of S approach. But we hope that some of the introduced notions will become useful when studying the Hodge-to-de-Rham spectral sequence.

Proposition 3.2.4. *Let n be a positive integer. Let S, T be finite truncation sets such that $S/n = \{1\} = T/n$. Let X be a smooth scheme over $\mathbb{Z}$. The following holds.*

1. *The morphism $\mathbb{W}_S\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_{S\backslash\{n\}}\Omega^q_{X/\mathbb{Z}}$ is surjective for all q.*
2. *If $T \subset S$ then the natural map*
$$\ker(\mathbb{W}_S\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_{S\backslash\{n\}}\Omega^q_{X/\mathbb{Z}}) \to \ker(\mathbb{W}_T\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_{T\backslash\{n\}}\Omega^q_{X/\mathbb{Z}})$$
is an isomorphism.

Proof. Note that $S/n = \{1\}$ is equivalent to $S\backslash\{n\}$ is a truncation set.

If $A \to B$ is an étale ring homomorphism then
$$\mathbb{W}_S B \otimes_{\mathbb{W}_S A} \mathbb{W}_{S\backslash\{n\}} A \xrightarrow{\cong} \mathbb{W}_{S\backslash\{n\}} B,$$
by Lemma 1.0.15. So that we can reduce in view of Lemma 2.2.6 to the following statements:

1. The map
$$\mathbb{W}_S\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}} \to \mathbb{W}_{S\backslash\{n\}}\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}}$$
is surjective.
2. The map
$$\ker(\mathbb{W}_S\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}} \to \mathbb{W}_{S\backslash\{n\}}\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}}) \to \ker(\mathbb{W}_T\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}} \to \mathbb{W}_{T\backslash\{n\}}\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}})$$
is an isomorphism.

After fixing a λ-structure on $\mathbb{Z}[x_1, \ldots, x_d]$ and by using Theorem 2.2.12 we have $\mathbb{W}_S\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}} = \prod_{k\in S}\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}}(k)$ and both statements become obvious. The module which is defined by (2) is simply $\Omega^q_{\mathbb{Z}[x_1,\ldots,x_d]/\mathbb{Z}}(n)$. □

In the following we suppose that X is a smooth scheme over $\mathbb{Z}$.

Definition 3.2.5. Let n be a positive integer. For all q, we define
$$\mathbb{W}_{\{n\}}\Omega^q_{X/\mathbb{Z}} := \ker(\mathbb{W}_S\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_{S\backslash\{n\}}\Omega^q_{X/\mathbb{Z}}),$$
where S is a finite truncation set with $S/n = \{1\}$.

In view of Proposition 3.2.4 the definition is independent of the choice of S. Moreover, we obtain a short exact sequence:
$$0 \to \mathbb{W}_{\{n\}}\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_{S\backslash\{n\}}\Omega^q_{X/\mathbb{Z}} \to 0. \tag{3.2.1}$$

It follows that if $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale then
$$\Gamma(\mathrm{Spec}(B), \mathbb{W}_{\{n\}}\Omega^q_{\mathrm{Spec}(B)/\mathbb{Z}}) = \Gamma(\mathrm{Spec}(A), \mathbb{W}_{\{n\}}\Omega^q_{\mathrm{Spec}(A)/\mathbb{Z}}) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S(B). \tag{3.2.2}$$

3.2.6. The morphism

$$V_n : \Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{X/\mathbb{Z}}$$

factors through $\mathbb{W}_{\{n\}}\Omega^q_{X/\mathbb{Z}}$ and we define

$$Q^q_{n,X} := \operatorname{coker}(\Omega^q_{X/\mathbb{Z}} \xrightarrow{V_n} \mathbb{W}_{\{n\}}\Omega^q_{X/\mathbb{Z}}).$$

Note that if we let $\mathbb{W}_S(\mathcal{O}_X)$ act on $\Omega^q_{X/\mathbb{Z}}$ via F_n, then we get an exact sequence of $\mathbb{W}_S(\mathcal{O}_X)$-modules

$$0 \to \Omega^q_{X/\mathbb{Z}} \xrightarrow{V_n} \mathbb{W}_{\{n\}}\Omega^q_{X/\mathbb{Z}} \to Q^q_{n,X} \to 0. \tag{3.2.3}$$

Again this implies

$$\Gamma(\operatorname{Spec}(B), Q^q_{n,\operatorname{Spec}(B)}) = \Gamma(\operatorname{Spec}(A), Q^q_{n,\operatorname{Spec}(A)}) \otimes_{\mathbb{W}_S(A)} \mathbb{W}_S(B), \tag{3.2.4}$$

provided that $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is étale.

Obviously, $Q^0_{n,X} = 0$ for all n. If A is a λ-ring then Theorem 2.2.12 yields

$$\Gamma(\operatorname{Spec}(A), Q^q_{n,\operatorname{Spec}(A)}) = \frac{d\Omega^{q-1}_{A/\mathbb{Z}}}{n\Omega^q_{A/\mathbb{Z}} \cap d\Omega^{q-1}_{A/\mathbb{Z}}}. \tag{3.2.5}$$

In particular, $Q^q_{n,X}$ is an n-torsion sheaf.

Lemma 3.2.7. *Let m, n be positive integers such that $m \mid n$. Let q be a non-negative integer. The morphism*

$$Q^q_{m,X} \xrightarrow{V_{\frac{n}{m}}} Q^q_{n,X}$$

is injective and has $\frac{n}{m}Q^q_{n,X}$ as image. In particular, if $n = \prod_i p_i^{\nu_i}$ is the prime decomposition then

$$Q^q_{n,X} \cong \prod_i Q^q_{p_i^{\nu_i}}.$$

Proof. By using base change for étale morphisms (3.2.4) this follows immediately from (3.2.5). □

3.2.8. The morphism

$$\Omega^{q-1}_{X/\mathbb{Z}} \xrightarrow{dV_n} Q^q_{n,X} \tag{3.2.6}$$

is compatible with the $\mathbb{W}_S(\mathcal{O}_X)$-module structures, because

$$\begin{aligned} dV_n(F_n(a)\omega) &= d(aV_n(\omega)) \\ &= da \cdot V_n(\omega) + a \cdot dV_n(\omega) \\ &= V_n(F_n(da)\omega) + a \cdot dV_n(\omega) \\ &= a \cdot dV_n(\omega) \mod V_n(\Omega^q_{X/\mathbb{Z}}). \end{aligned}$$

Therefore we can show by reduction via étale maps and (3.2.5) that (3.2.6) is surjective. In particular, we obtain a surjective morphism

$$\tau_{p^n} : \Omega^{q-1}_{X\otimes\mathbb{F}_p} \to Q^q_{p^n,X}/pQ^q_{p^n,X}.$$

Proposition 3.2.9. *Suppose that X is a smooth scheme over $\mathbb{Z}$. Let p be a prime. For all positive integers n, q the sequence*

$$0 \to \Omega^{q-1}_{X\otimes\mathbb{F}_p} \xrightarrow{C^{-n}} \Omega^{q-1}_{X\otimes\mathbb{F}_p}/B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p} \xrightarrow{\tau_{p^n}} Q^q_{p^n,X}/pQ^q_{p^n,X} \to 0$$

is exact.

Proof. Recall from [Ill79, p.519] that we have the inverse Cartier operator at our disposal:

$$C^{-1} : \Omega^{q-1}_{X\otimes\mathbb{F}_p} \to \Omega^{q-1}_{X\otimes\mathbb{F}_p}/d\Omega^{q-2}_{X\otimes\mathbb{F}_p}.$$

The subsheaf $B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p}$ of $\Omega^{q-1}_{X\otimes\mathbb{F}_p}$ is defined by

$$B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p} = \sum_{i=0}^{n-1} C^{-i}(d\Omega^{q-2}_{X\otimes\mathbb{F}_p}).$$

(cf. [Ill79, p.519]). If we let $\mathcal{O}_{X\otimes\mathbb{F}_p}$ act on $\Omega^{q-1}_{X\otimes\mathbb{F}_p}$ via Frob^n, where Frob is the absolute Frobenius, then $B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p}$ is a locally free $\mathcal{O}_X$-module of finite rank [Ill79, Proposition 2.2.8]. We have an injective map

$$\Omega^{q-1}_{X\otimes\mathbb{F}_p} \xrightarrow{C^{-n}} \Omega^{q-1}_{X\otimes\mathbb{F}_p}/B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p} \tag{3.2.7}$$

that has $Z_n\Omega^{q-1}_{X\otimes\mathbb{F}_p}/B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p}$ as image [Ill79, (2.2.5)].

Set $S = \{1, p, p^2, \ldots, p^n\}$; the $\mathbb{W}_S(\mathcal{O}_X)$-module structure on the source of τ_{p^n} is by construction induced via $\mathbb{W}_S(\mathcal{O}_X) \xrightarrow{F^n_p} \mathcal{O}_X \to \mathcal{O}_{X\otimes\mathbb{F}_p}$ which equals $\mathbb{W}_S(\mathcal{O}_X) \xrightarrow{\mathrm{proj}} \mathcal{O}_X \to \mathcal{O}_{X\otimes\mathbb{F}_p} \xrightarrow{\mathrm{Frob}^n} \mathcal{O}_{X\otimes\mathbb{F}_p}$. Therefore $B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p}$ is a $\mathbb{W}_S(\mathcal{O}_X)$-submodule. Giving the source of (3.2.7) the $\mathbb{W}_S(\mathcal{O}_X)$-module structure via $\mathbb{W}_S(\mathcal{O}_X) \xrightarrow{\mathrm{proj}} \mathcal{O}_X \to \mathcal{O}_{X\otimes\mathbb{F}_p}$ makes (3.2.7) to a morphism of $\mathbb{W}_S(\mathcal{O}_X)$-modules.

Certainly it suffices to prove that for all $X = \mathrm{Spec}(B)$ that admit an étale morphism $X \to \mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_d])$ the following holds:

- $\tau_{p^n} : \Omega^{q-1}_{B\otimes\mathbb{F}_p} \to \Gamma(X, Q^q_{p^n,X})/p\Gamma(X, Q^q_{p^n,X})$ factors through $\Gamma(X, B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p})$,
- the following sequence is exact:

$$0 \to \Omega^{q-1}_{B\otimes\mathbb{F}_p} \xrightarrow{C^{-n}} \Omega^{q-1}_{B\otimes\mathbb{F}_p}/\Gamma(X, B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p}) \xrightarrow{\tau_{p^n}} \Gamma(X, Q^q_{p^n,X})/p\Gamma(X, Q^q_{p^n,X}) \to 0$$

Set $A = \mathbb{Z}[x_1, \ldots, x_d]$; in order to reduce both statements to A we need (3.2.4) and

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A),\mathrm{proj}} \Omega^{q-1}_{A\otimes\mathbb{F}_p} \cong \Omega^{q-1}_{B\otimes\mathbb{F}_p} \tag{3.2.8}$$

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A),F^n_p} \Omega^{q-1}_{A\otimes\mathbb{F}_p} \cong \Omega^{q-1}_{B\otimes\mathbb{F}_p} \tag{3.2.9}$$

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A)} \Gamma(\mathrm{Spec}(A), B_n\Omega^{q-1}_{\mathrm{Spec}(A\otimes\mathbb{F}_p)}) \cong \Gamma(\mathrm{Spec}(B), B_n\Omega^{q-1}_{X\otimes\mathbb{F}_p}). \tag{3.2.10}$$

The isomorphism (3.2.8) follows from $\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A),\mathrm{proj}} A \cong B$ (Lemma 1.0.15). The isomorphism (3.2.9) follows from $\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A),F^n_p} A \cong B$ (Theorem 1.0.14). Finally, the isomorphism (3.2.10) follows from

$$\mathbb{W}_S(B) \otimes_{\mathbb{W}_S(A),F^n_p} M \cong B/p \otimes_{A/p,\mathrm{Frob}^n} M, \quad b \otimes m \mapsto \mathrm{proj}(b) \otimes m,$$

for every A/p-module M, and [Ill79, (2.2.7)]. Thus we may assume $A = B = \mathbb{Z}[x_1, \ldots, x_d]$ and we can choose a λ-structure.

Using (3.2.5) the map τ_{p^n} is given by

$$\Omega^{q-1}_{A\otimes\mathbb{F}_p} \to \frac{d\Omega^{q-1}_{A/\mathbb{Z}}}{pd\Omega^{q-1}_{A/\mathbb{Z}} + \left(p^n\Omega^q_{A/\mathbb{Z}} \cap d\Omega^{q-1}_{A/\mathbb{Z}}\right)}, \quad \omega \mapsto d\tilde{\omega},$$

where $\tilde{\omega}$ is some lifting of ω. We have

$$\Gamma(\mathrm{Spec}(A), B_n\Omega^{q-1}_{\mathrm{Spec}(A\otimes\mathbb{F}_p)}) = \mathrm{image}(\sum_{i=0}^{n-1} F_p^i d\Omega^{q-2}_{A/\mathbb{Z}})$$

$$\Gamma(\mathrm{Spec}(A), Z_n\Omega^{q-1}_{\mathrm{Spec}(A\otimes\mathbb{F}_p)}) = \mathrm{image}(F_p^n\Omega^{q-1}_{A/\mathbb{Z}}) + \mathrm{image}(\sum_{i=0}^{n-1} F_p^i d\Omega^{q-2}_{A/\mathbb{Z}}).$$

Therefore the statement is implied by Lemma 2.1.5. □

Proof of Theorem 3.2.2. Note that $\mathbb{W}_S(R)$ is a noetherian ring. We prove the claim by induction on the length of S; set $n = \max(S)$. By using the short exact sequence (3.2.1) we reduce to the same statement for $\mathbb{W}_{\{n\}}\Omega^j_{X/\mathbb{Z}}$. In view of the short exact sequence (3.2.3) and the surjectivity of $\mathbb{W}_S(R) \xrightarrow{F_n} R$ it suffices to prove the same statement for $Q^j_{n,X}$. Moreover, we used that $H^{\dim X}(X, \Omega^j_{X/\mathbb{Z}}) = 0$, because the relative dimension of X over R equals $\dim X - 1$.

Lemma 3.2.7 implies $Q^j_{n,X} \cong \prod_k Q^j_{p_k^{\nu_k},X}$, if $n = \prod_k p_k^{\nu_k}$, and the short exact sequence

$$0 \to Q^j_{p_k^{\nu_k-1},X} \xrightarrow{V_p} Q^j_{p_k^{\nu_k},X} \to Q^j_{p_k^{\nu_k},X}/pQ^j_{p_k^{\nu_k},X} \to 0.$$

Thus we are reduced to consider $Q^j_{p^\nu,X}/pQ^j_{p^\nu,X}$ with p a prime. Finally, Proposition 3.2.9 and the fact that $B_n\Omega^{j-1}_{X\otimes\mathbb{F}_p}$ are locally free $\mathcal{O}_{X\otimes\mathbb{F}_p}$-modules via the Frob^n-action imply the claim. □

Proposition 3.2.10. *Let R be an étale $\mathbb{Z}$-algebra. Let X be a smooth and proper scheme over R. Let S be a finite truncation set. Let $\{U_i\}_{i\in I}$ be a finite covering of X by affine open subschemes. We denote the Čech cohomology with respect to $\{U_i\}_{i\in I}$ by $H^*(\{U_i\}, -)$. The following holds.*

(i) *For all t, q:*
$$H^t(\{U_i\}, \mathbb{W}_S\Omega^q_{X/\mathbb{Z}}) = H^t(X, \mathbb{W}_S\Omega^q_{X/\mathbb{Z}}).$$
(ii) *For all t:*
$$H^t(\{U_i\}, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}}) = H^t(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}}).$$
(iii) *The natural morphism of complexes*
$$s\left(\oplus_{i_0}\mathbb{W}_S\Omega^*_{\Gamma(U_{i_0},\mathcal{O})/\mathbb{Z}} \to \oplus_{i_0<i_1}\mathbb{W}_S\Omega^*_{\Gamma(U_{i_0}\cap U_{i_1},\mathcal{O})/\mathbb{Z}} \to \dots\right) \to R\Gamma(\mathbb{W}_S\Omega^*_{X/\mathbb{Z}}),$$
where $s(-)$ is the associated simple complex, is a quasi-isomorphism.

Proof. Statement (i) can be proved by the same arguments that has been used in the proof of Theorem 3.2.2. The assertions (ii) and (iii) follow easily from (i). □

Corollary 3.2.11. *Let R be an étale $\mathbb{Z}$-algebra. Let X be a smooth and proper R-scheme. Let p be a prime, and let n, m be positive integers. There is a natural quasi-isomorphism of complexes of $W_n(R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$-modules:*

$$R\Gamma(W_n\Omega^*_{X/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \otimes^{\mathbb{L}}_{W_n(R\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)})} W_n(R/p^m) \to R\Gamma(W_n\Omega^*_{X\otimes_R R/p^m/(R/p^m)}).$$

Proof. In view of Theorem 3.1.2 the claim follows from Proposition 3.2.10(iii) and the analogous statement for the relative de Rham-Witt complex (over R/p^m) of Langer-Zink [LZ04, Section 1.4]. □

3.3. **Flatness.** The goal of this section is to prove the following theorem.

Theorem 3.3.1. *Let R be an étale $\mathbb{Z}$-algebra. Let X be a smooth and proper R-scheme. Suppose that the de Rham cohomology $H^*_{dR}(X/R)$ of X is a flat R-module. Then $H^*(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$ is a finitely generated projective $\mathbb{W}_S(R)$-module for all finite truncation sets S.*

Since $\mathbb{W}_S(R)$ is a noetherian ring and we know that $H^*(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$ is a finitely generated $\mathbb{W}_S(R)$-module (Corollary 3.2.3), it remains to show that it is flat. This is a local property and can be checked prime by prime. Our proof relies on Theorem 3.1.2 or, more precisely, Corollary 3.2.11.

Lemma 3.3.2. *Let R be an étale $\mathbb{Z}$-algebra. Let $\mathfrak{m}$ be a maximal ideal of R, and let n be a positive integer. Set $p = \mathrm{char}(R/\mathfrak{m})$. Then $W_n(R_{\mathfrak{m}}) \to W_n(\varprojlim_i R/\mathfrak{m}^i)$ is faithfully flat.*

Proof. Note that $\mathfrak{m}R_{\mathfrak{m}} = pR_{\mathfrak{m}}$. By Lemma 1.0.27, $W_n(R_{\mathfrak{m}})$ is a local ring with maximal ideal $\ker(W_n(R_{\mathfrak{m}}) \to R/\mathfrak{m})$. Moreover, $W_n(R_{\mathfrak{m}})$ is a noetherian ring.

We will show that $W_n(\varprojlim_i R/\mathfrak{m}^i)$ is the p-adic completion of $W_n(R_{\mathfrak{m}})$:

$$W_n(\varprojlim_i R/\mathfrak{m}^i) = \varprojlim_i W_n(R_{\mathfrak{m}})/p^i, \tag{3.3.1}$$

which proves the flatness. Furthermore, the maximal ideal of $W_n(R_{\mathfrak{m}})$ is the preimage of $\ker(W_n(\varprojlim_i R/\mathfrak{m}^i) \to \varprojlim_i R/\mathfrak{m}^i \to R/\mathfrak{m})$ for the ring homomorphism $W_n(R_{\mathfrak{m}}) \to W_n(\varprojlim_i R/\mathfrak{m}^i)$, hence the morphism will be faithfully flat.

In order to prove (3.3.1) we need to show that

$$0 \to R_{\mathfrak{m}}/p^i \xrightarrow{V_p^n} W_n(R_{\mathfrak{m}})/p^i \to W_{n-1}(R_{\mathfrak{m}})/p^i \to 0 \tag{3.3.2}$$

is an exact sequence for all i. It remains exact after taking $\varprojlim_i$, because $R_{\mathfrak{m}}/p^i$ is a finite set. Therefore (3.3.1) will follow by induction on n.

Suppose $x \in R_{\mathfrak{m}}$ satisfies $V_p^n(x) = p^i y$ for some $y \in W_n(R_{\mathfrak{m}})$. Since $W_{n-1}(R_{\mathfrak{m}})$ is p-torsion free, we get $y = V_p^n(z)$, hence $x = p^i z$. This shows the exactness of (3.3.2). □

Lemma 3.3.3. *Let R be an étale $\mathbb{Z}$-algebra. Let $\mathfrak{m}$ be a maximal ideal of R, let n be a positive integer, and set $p = \mathrm{char}(R/\mathfrak{m})$. Let C be a bounded complex of $W_n(R_{\mathfrak{m}})$-modules such that $H^i(C)$ is a finitely generated $W_n(R_{\mathfrak{m}})$-module for all i. Then, for all i,*

$$H^i(C) \otimes_{W_n(R_{\mathfrak{m}})} W_n(\varprojlim_j R/\mathfrak{m}^j) \cong \varprojlim_j H^i\left(C \otimes^{\mathbb{L}}_{W_n(R_{\mathfrak{m}})} W_n(R/\mathfrak{m}^j)\right).$$

Proof. Set $\hat{R} := \varprojlim_j R/\mathfrak{m}^j$. The map is induced by $C \to C \otimes^{\mathbb{L}}_{W_n(R_\mathfrak{m})} W_n(R/\mathfrak{m}^j)$ and the $W_n(\hat{R})$-module structure on the right hand side.

As a first step we will prove that $H^i\left(C \otimes^{\mathbb{L}}_{W_n(R_\mathfrak{m})} W_n(R/\mathfrak{m}^j)\right)$ is a finite group. Clearly, we may assume that $C = C_0$ is concentrated in degree 0. Since C_0 is finitely generated we conclude that $\mathrm{Tor}^i_{W_n(R_\mathfrak{m})}(C_0, W_n(R/\mathfrak{m}^j))$ is a finitely generated $W_n(R/\mathfrak{m}^j)$-module for all i. The ring $W_n(R/\mathfrak{m}^j)$ contains only finitely many elements, hence

$$H^{-i}(C \otimes^{\mathbb{L}}_{W_n(R_\mathfrak{m})} W_n(R/\mathfrak{m}^j)) = \mathrm{Tor}_i(C_0, W_n(R/\mathfrak{m}^j))$$

is finite.

By using Lemma 3.3.2 and the first step (all $R^1 \varprojlim$ vanish) we can reduce the assertion to the case of a complex $C = C_0$ that is concentrated in degree zero (hence C_0 is finitely generated). In this case we need to show:

(a) $C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(\hat{R}) \xrightarrow{=} \varprojlim_j (C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(R/\mathfrak{m}^j))$,
(b) $\varprojlim_j \mathrm{Tor}_i(C_0, W_n(R/\mathfrak{m}^j)) = 0$ for all $i > 0$.

For claim (a) we use that $W_n(\hat{R}) = \varprojlim_j W_n(R_\mathfrak{m})/p^j$, which has been already used in the proof of Lemma 3.3.2. The natural map

$$C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(R_\mathfrak{m})/p^{jn} \to C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(R/\mathfrak{m}^j)$$

is surjective and yields a surjective map

$$\varprojlim_j (C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(R_\mathfrak{m})/p^j) \to \varprojlim_j (C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(R/\mathfrak{m}^j))$$

(again the $R^1 \varprojlim$ vanish). Since $C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(\hat{R}) = \varprojlim_j (C_0 \otimes_{W_n(R_\mathfrak{m})} W_n(R_\mathfrak{m})/p^j)$, we conclude that the map in claim (a) is surjective. Then it is easy to prove that it is an isomorphism by using the flatness of $W_n(R_\mathfrak{m}) \to W_n(\hat{R})$.

Claim (b) follows from (a) and the flatness of $W_n(R_\mathfrak{m}) \to W_n(\hat{R})$. $\square$

Proposition 3.3.4. *Assumptions as in Theorem 3.3.1. Let $\mathfrak{m}$ be a maximal ideal of R, let n be a positive integer, and set $p = \mathrm{char}(R/\mathfrak{m})$. For all i, we have an isomorphism*

$$(3.3.3) \quad H^i(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Z}_p) \otimes_{W_n(R\otimes\mathbb{Z}_{(p)})} W_n(\varprojlim_j R/\mathfrak{m}^j) \xrightarrow{\cong} \varprojlim_j H^i(X \otimes_R R/\mathfrak{m}^j, W_n\Omega^*_{X\otimes_R(R/\mathfrak{m}^j)/(R/\mathfrak{m}^j)}).$$

Proof. Let $U \to \mathrm{Spec}(R)$ be an open subscheme, set $R_U := \Gamma(U, \mathcal{O})$. If $\mathfrak{m} \in U$ then the target of (3.3.3) is the same for R and R_U. For the source we have to use Lemma 2.2.6 and Corollary 1.0.16 in order to see that it is also the same for R and R_U. Therefore we may remove all points of $\mathrm{Spec}(R)$ that are different from $\mathfrak{m}$ and are lying over p. In other words, we may assume that $\mathfrak{m} = pR$.

In view of Corollary 3.2.3 and Corollary 3.2.11 the claim follows from Lemma 3.3.3. $\square$

Corollary 3.3.5. *Assumption as in Proposition 3.3.4. For all i we have an isomorphism*

$$H^i(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \otimes_{W_n(R\otimes\mathbb{Z}_{(p)})} W_n(\varprojlim_j R/\mathfrak{m}^j) \xrightarrow{\cong} H^i_{dR}(X \otimes_R R_{\mathfrak{m}}/R_{\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} W_n(\varprojlim_j R/\mathfrak{m}^j).$$

Proof. Although in general there is no ring homomorphism $R_{\mathfrak{m}} \to W_n(R_{\mathfrak{m}})$, note that we have a ring homomorphism $R_{\mathfrak{m}} \to W_n(\varprojlim_j R/\mathfrak{m}^j)$ induced by the Frobenius lifting on $\varprojlim_j R/\mathfrak{m}^j$.

As in the proof of Proposition 3.3.4, we may assume that $\mathfrak{m} = pR$. We set $R' = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. We use Proposition 3.3.4 to rewrite the left hand side as a projective limit. As explained in Section 3.1.4 we have a comparison isomorphism with de Rham cohomology

$$H^i(X \otimes_R R/p^j, W_n\Omega^*_{X\otimes_R R/p^j/(R/p^j)}) \cong H^i_{dR}(X \otimes_R R'/R') \otimes_{R'} W_n(R/p^j),$$

which yields

$$\varprojlim_j H^i(X \otimes_R R/p^j, W_n\Omega^*_{X\otimes_R R/p^j/(R/p^j)}) \cong H^i_{dR}(X \otimes_R R'/R') \otimes_{R'} W_n(\varprojlim_j R/p^j),$$

because the de Rham cohomology is finitely generated. □

Proof of Theorem 3.3.1. Without loss of generality we may assume that R is integral. It suffices to show the flatness of $H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$ when considered as a $\mathbb{W}_S(R)$-module. This can be checked after localizing at maximal ideals. By Lemma 1.0.26 it is sufficient to show that $H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}}) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R \otimes \mathbb{Z}_{(p)})$ is a flat $\mathbb{W}_S(R \otimes \mathbb{Z}_{(p)})$-module for all primes p such that $pR \neq R$. Recall that $\mathbb{W}_S(R \otimes \mathbb{Z}_{(p)}) = \mathbb{W}_S(R) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ (Lemma 1.0.9), hence

$$H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}}) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R \otimes \mathbb{Z}_{(p)}) = H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}} \otimes \mathbb{Z}_{(p)}).$$

By using the decomposition of $\mathbb{W}_S\Omega^*_{X/\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$ from Proposition 2.3.1 together with Corollary 1.0.22 we may assume that S is p-typical, say $S = \{1, p, \dots, p^{n-1}\}$.

Since R is integral and $pR \neq R$, every maximal ideal $\mathfrak{a}$ of $W_n(R \otimes \mathbb{Z}_{(p)})$ is the form

$$\mathfrak{a} = \ker(W_n(R \otimes \mathbb{Z}_{(p)}) \xrightarrow{\pi_{\{1\}}} R \otimes \mathbb{Z}_{(p)} \to R/\mathfrak{m}),$$

for a maximal ideal $\mathfrak{m}$ of R, thus $W_n(R \otimes \mathbb{Z}_{(p)})_{\mathfrak{a}} = W_n(R_{\mathfrak{m}})$ (Lemma 1.0.27).

By assumption, $H^i_{dR}(X \otimes_R R_{\mathfrak{m}}/R_{\mathfrak{m}}) = H^i_{dR}(X/R) \otimes_R R_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module. Therefore Lemma 3.3.2 and Corollary 3.3.5 imply the claim. □

Corollary 3.3.6. *Assumptions as in Theorem 3.3.1. Let $\mathfrak{m}$ be a maximal ideal of R, and set $p = \mathrm{char}(R/\mathfrak{m})$. Let n, j be positive integers. The natural map*

$$H^i(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \otimes_{W_n(R\otimes\mathbb{Z}_{(p)})} W_n(R/\mathfrak{m}^j) \to H^i(X \otimes_R (R/\mathfrak{m}^j), W_n\Omega^*_{X\otimes_R(R/\mathfrak{m}^j)/(R/\mathfrak{m}^j)})$$

is an isomorphism for all i.

Proof. As in the proof of Proposition 3.3.4 we may assume that $\mathfrak{m} = pR$. The claim follows immediately from Corollary 3.2.11 and Theorem 3.3.1. □

Proposition 3.3.7. *With the assumptions as in Theorem 3.3.1. If $T \subset S$ is an inclusion of finite truncation sets then*

$$\mathbb{W}_T(R) \otimes_{\mathbb{W}_S(R)} H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}}) \to H^i(X, \mathbb{W}_T\Omega^*_{X/\mathbb{Z}})$$

is an isomorphism for all i.

Proof. Arguing as in the proof of Theorem 3.3.1, we have to show that

$$\begin{aligned}(3.3.4)\quad \mathbb{W}_T(R_\mathfrak{m}) \otimes_{\mathbb{W}_S(R\otimes\mathbb{Z}_{(p)})} H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Z}_{(p)}) \\ \to \mathbb{W}_T(R_\mathfrak{m}) \otimes_{\mathbb{W}_T(R\otimes\mathbb{Z}_{(p)})} H^i(X, \mathbb{W}_T\Omega^*_{X/\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Z}_{(p)})\end{aligned}$$

is an isomorphism for every maximal ideal $\mathfrak{m}$ of R and $(p) = \mathfrak{m} \cap \mathbb{Z}$. Again, we can reduce to the case where S is p-typical.

Now, $\mathbb{W}_T(R_\mathfrak{m})$ is a local ring, hence (3.3.4) is a morphism of free modules of the same rank, that is $\operatorname{rk} H^i_{dR}(X \otimes R_\mathfrak{m}/R_\mathfrak{m})$ (Theorem 3.3.1, Corollary 3.3.5). Therefore it is enough to prove the surjectivity of (3.3.4) modulo the maximal ideal. Via Corollary 3.3.5 both sides agree with $H^i_{dR}(X \otimes_R (R/\mathfrak{m})/(R/\mathfrak{m}))$ modulo the maximal ideal, and the map is simply the identity. □

3.3.8. For all positive integers n and all finite truncation sets we set

$$\phi_n = n^q F_n : \mathbb{W}_S\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_{S/n}\Omega^q_{X/\mathbb{Z}}, \tag{3.3.5}$$

to get a morphism of complexes

$$\mathbb{W}_S\Omega^*_{X/\mathbb{Z}} \xrightarrow{\phi_n} \mathbb{W}_{S/n}\Omega^*_{X/\mathbb{Z}}.$$

Suppose that $\dim X = d + 1$. Then we set

$$\beta_n = n^{d-q} V_n : \mathbb{W}_{S/n}\Omega^q_{X/\mathbb{Z}} \to \mathbb{W}_S\Omega^q_{X/\mathbb{Z}}$$

(note that $\mathbb{W}_S\Omega^q_{X/\mathbb{Z}} = 0$ if $q > d$, because this vanishing result holds for $X = \mathbb{A}^d_\mathbb{Z}$ in view of Theorem 2.2.12). We obtain a morphism of complexes

$$\beta_n : \mathbb{W}_{S/n}\Omega^*_{X/\mathbb{Z}} \to \mathbb{W}_S\Omega^*_{X/\mathbb{Z}},$$

satisfying the equalities:

$$\begin{aligned}&\phi_n \circ \beta_n = n^{d+1},\\ &\beta_n(\lambda \cdot \phi_n(x)) = n^d V_n(\lambda) \cdot x \quad \text{for all } x \in \mathbb{W}_S\Omega^*_{X/\mathbb{Z}} \text{ and } \lambda \in \mathbb{W}_{S/n}\Omega^*_{X/\mathbb{Z}}.\end{aligned}$$

We will study the $\{\phi_n\}_{n\geq 1}$ operations induced on the de Rham-Witt cohomology

$$S \mapsto H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}}),$$

for a smooth and proper scheme X over $\mathbb{Z}[N^{-1}]$ in Section 4.

3.3.9. Let X be an R-scheme such that the assumptions of Theorem 3.3.1 are satisfied. Fix a maximal ideal $\mathfrak{m}$ of R. Again, define the prime p by $(p) = \mathfrak{m} \cap R$.

On the one hand we have the $W(R\otimes\mathbb{Z}_{(p)})$-module $\varprojlim_n H^*(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ together with the ϕ_p endomorphism induced by (3.3.5). On the other hand we have the classical de Rham-Witt cohomology $\varprojlim_n H^*(X\otimes_R R/\mathfrak{m}, W_n\Omega^*_{X\otimes_R(R/\mathfrak{m})/(R/\mathfrak{m})})$, with its ϕ_p-action, which only depends on the fibre of X over $R/\mathfrak{m}$, and is a $W(R/\mathfrak{m})$-module.

It is important for us to compare this modules together with the ϕ_p-actions, and the next lemma shows that after passing from $R_\mathfrak{m}$ to the completion $\hat{R} := \varprojlim_n R/\mathfrak{m}^n$, this can be done in an evident way.

Lemma 3.3.10. *For all i, there is an isomorphism*

$$\begin{aligned}(3.3.6) \qquad \varprojlim_n H^i(X\otimes_R R/\mathfrak{m}, W_n\Omega^*_{X\otimes_R(R/\mathfrak{m})/(R/\mathfrak{m})}) \otimes_{W(R/\mathfrak{m})} W(\hat{R}) \\ \to \varprojlim_n H^i(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \otimes_{W(R\otimes\mathbb{Z}_{(p)})} W(\hat{R}),\end{aligned}$$

which is compatible with the $\phi_p \otimes F_p$-endomorphisms on both sides.

Proof. Without loss of generality we may assume that R is integral. Set $r := \mathrm{rank}(H^i_{dR}(X/R))$, $k := R/\mathfrak{m}$, and $X_0 := X \otimes_R k$

By Theorem 3.3.1 and Corollary 3.3.6 we conclude that $H^i(X_0, W_n\Omega^*_{X_0/k})$ is a free $W_n(k)$-module of rank r. Every basis of $H^i(X_0, W_n\Omega^*_{X_0/k})$ lifts to a basis of $H^i(X_0, W_{n+1}\Omega^*_{X_0/k})$, hence $\varprojlim_n H^i(X_0, W_n\Omega^*_{X_0/k})$ is a free $W(k)$-module with

$$\varprojlim_n H^i(X_0, W_n\Omega^*_{X_0/k}) \otimes_{W(k)} k = H^i(X_0, \Omega^*_{X_0/k}).$$

The ring $W(\hat{R})$ is a local ring with maximal ideal $\mathfrak{a} := \ker(W(\hat{R}) \xrightarrow{\pi_{\{1\}}} \hat{R} \to k)$. The left hand side of (3.3.6) is a free $W(\hat{R})$-module of rank r with reduction $H^i(X_0, \Omega^*_{X_0/k})$ modulo $\mathfrak{a}$.

Similarly, by using Lemma 1.0.27, we conclude that $\varprojlim_n H^i(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ is a free $W(R\otimes\mathbb{Z}_{(p)})$-module of rank r, and the reduction of the right hand side of (3.3.6) modulo $\mathfrak{a}$ is given by $H^i(X, \Omega^*_{X/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \otimes_{R\otimes\mathbb{Z}_{(p)}} k = H^i(X_0, \Omega^*_{X_0/k})$.

Therefore it suffices to construct the map (3.3.6), to show that it is an isomorphism modulo $\mathfrak{a}$, and to check the compatibility with $\phi_p \otimes F_p$.

Clearly, we have $W(\hat{R}) = \varprojlim_n W_n(R/\mathfrak{m}^n)$, and it follows easily that the left hand side of (3.3.6) is the limit

$$\varprojlim_n \left(H^i(X_0, W_{n^2}\Omega^*_{X_0/k}) \otimes_{W_{n^2}(k)} W_n(R/\mathfrak{m}^n) \right),$$

whereas the right hand side is the limit

$$\varprojlim_n \left(H^i(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes \mathbb{Z}_{(p)}) \otimes_{W_n(R\otimes\mathbb{Z}_{(p)})} W_n(R/\mathfrak{m}^n) \right).$$

From now on, we will use the notation $X_{n-1} := X \otimes_R R/\mathfrak{m}^n$ and identify

$$H^i(X, W_n\Omega^*_{X/\mathbb{Z}} \otimes \mathbb{Z}_{(p)}) \otimes_{W_n(R\otimes\mathbb{Z}_{(p)})} W_n(R/\mathfrak{m}^n) \cong H^i(X_{n-1}, W_n\Omega^*_{X_{n-1}/(R/\mathfrak{m}^n)})$$

(Corollary 3.3.6). The comparison isomorphism [LZ04, Theorem 3.5] yields

$$H^i(X_0, W_{n^2}\Omega^*_{X_0/k}) \cong H^i_{crys}(X_0/W_{n^2}(k)), \tag{3.3.7}$$

$$H^i(X_{n-1}, W_n\Omega^*_{X_{n-1}/(R/\mathfrak{m}^n)}) \cong H^i_{crys}(X_{n-1}/W_n(R/\mathfrak{m}^n)). \tag{3.3.8}$$

In (3.3.8) the crystalline cohomology is taken with respect to the pd-ideal $I := \ker(W_n(R/\mathfrak{m}^n) \xrightarrow{\pi_{\{1\}}} R/\mathfrak{m}^n)$, in (3.3.7) with respect to $pW_{n^2}(k)$. In order to avoid confusion we will use the notation $H^i_{crys}(X_{n-1}/(W_n(R/\mathfrak{m}^n), I))$ in the following.

The ring homomorphism $W_{n^2}(k) \to W_n(R/\mathfrak{m}^n)$, which is induced by the Frobenius on $W(k)$, yields a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "id"] \arrow[d] & X_0 \arrow[d] \\ \mathrm{Spec}(W_n(R/\mathfrak{m}^n)) \arrow[r] & \mathrm{Spec}(W_{n^2(k)}). \end{tikzcd} \tag{3.3.9}$$

We obtain a map

$$\begin{aligned} H^i_{crys}(X_0/W_{n^2}(k)) &\to H^i_{crys}(X_0/(W_n(R/\mathfrak{m}^n), I+(p))) \\ &\xrightarrow{\cong} H^i_{crys}(X_{n-1}/(W_n(R/\mathfrak{m}^n), I+(p))), \end{aligned} \tag{3.3.10}$$

where we have used [BO78, Theorem 5.17] for the last isomorphism. The pd-ideal $I + (p)$ is an extension of I and (p). The evident morphism

$$\eta : H^i_{crys}(X_{n-1}/(W_n(R/\mathfrak{m}^n), I)) \to H^i_{crys}(X_{n-1}/(W_n(R/\mathfrak{m}^n), I+(p)))$$

is an isomorphism, because both cohomologies identify via the comparison with de Rham cohomology to $H^i_{dR}(X \otimes_R W_n(R/\mathfrak{m}^n)/W_n(R/\mathfrak{m}^n))$. The composition of (3.3.10) with η^{-1} yields the desired map

$$H^i(X_0, W_{n^2}\Omega^*_{X_0/k}) \otimes_{W_{n^2}(k)} W_n(R/\mathfrak{m}^n) \to H^i(X_{n-1}, W_n\Omega^*_{X_{n-1}/(R/\mathfrak{m}^n)}).$$

After passing to the limit we obtain (3.3.6). Modulo $\mathfrak{a}$, it is simply the identity, hence it remains to prove the compatibility with ϕ_p.

The commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{Frob}"] \arrow[d] & X_0 \arrow[d] \\ \mathrm{Spec}(W_{n-1}(R/\mathfrak{m}^n)) \arrow[r, "F_p"] & \mathrm{Spec}(W_n(R/\mathfrak{m}^n)) \end{tikzcd}$$

induces a morphism

$$\mathrm{Frob}^* : H^i_{crys}(X_0/(W_n(R/\mathfrak{m}^n), I+(p))) \to H^i_{crys}(X_0/(W_{n-1}(R/\mathfrak{m}^n), I+(p))).$$

Obviously, the diagram
(3.3.11)

$$\begin{tikzcd}
H^i_{crys}(X_0/W_{n^2}(k)) \arrow[r, "\text{via (3.3.9)}"] \arrow[d, "\mathrm{Frob}^*"] & H^i_{crys}(X_0/(W_n(R/\mathfrak{m}^n), I+(p))) \arrow[d, "\mathrm{Frob}^*"] \\
H^i_{crys}(X_0/W_{n^2}(k)) \arrow[r] \arrow[d] & H^i_{crys}(X_0/(W_{n-1}(R/\mathfrak{m}^n), I+(p))) \arrow[d] \\
H^i_{crys}(X_0/W_{(n-1)^2}(k)) \arrow[r, "\text{via (3.3.9)}"] & H^i_{crys}(X_0/(W_{n-1}(R/\mathfrak{m}^{n-1}), I+(p))),
\end{tikzcd}$$

is commutative. Moreover, [LZ04, Proposition 3.6] implies that

$$\begin{tikzcd}
H^i_{crys}(X_0/(W_n(R/\mathfrak{m}^n), I+(p))) \arrow[r, "\cong"] \arrow[d, "\mathrm{Frob}^*"] & H^i(X_{n-1}, W_n\Omega^*_{X_{n-1}/(R/\mathfrak{m}^n)}) \arrow[d, "\phi_p"] \\
H^i_{crys}(X_0/(W_{n-1}(R/\mathfrak{m}^n), I+(p))) \arrow[r, "\cong"] \arrow[d] & H^i(X_{n-1}, W_{n-1}\Omega^*_{X_{n-1}/(R/\mathfrak{m}^n)}) \arrow[d] \\
H^i_{crys}(X_0/(W_{n-1}(R/\mathfrak{m}^{n-1}), I+(p))) \arrow[r, "\cong"] & H^i(X_{n-2}, W_{n-1}\Omega^*_{X_{n-2}/(R/\mathfrak{m}^{n-1})})
\end{tikzcd}$$

is commutative. Furthermore, we have a similar commutative diagram for the left column of (3.3.11) and the de Rham-Witt cohomology of X_0. This completes the proof. $\square$

4. Values of the big de Rham-Witt cohomology

4.1. **A rigid $\otimes$-category.**

Definition 4.1.1. Let R be an étale $\mathbb{Z}$-algebra or $\mathbb{Z}_{(p)}$-algebra, where p is a prime. We denote by $\mathcal{C}'_R$ the category with objects being contravariant functors $S \mapsto M_S$ from finite truncation sets to sets together with

- a $\mathbb{W}_S(R)$-module structure on M_S, for all truncation sets S, such that the maps $M_S \to M_T$, for $T \subset S$, are morphisms of $\mathbb{W}_S(R)$-modules when M_T is considered as a $\mathbb{W}_S(R)$-module via the projection $\pi_T : \mathbb{W}_S(R) \to \mathbb{W}_T(R)$,
- for all positive integers n and all truncation sets S, maps

$$\phi_n : M_S \to M_{S/n},$$

  such that
  - $\phi_n \circ \phi_m = \phi_{nm}$ for all n, m,
  - ϕ_n is a morphism of $\mathbb{W}_S(R)$-modules when $M_{S/n}$ is considered as a $\mathbb{W}_S(R)$ module via $F_n : \mathbb{W}_S(R) \to \mathbb{W}_{S/n}(R)$,
  - for all truncation sets $T \subset S$ the following diagram is commutative:

$$\begin{tikzcd}
M_S \arrow[r, "\phi_n"] \arrow[d] & M_{S/n} \arrow[d] \\
M_T \arrow[r, "\phi_n"] & M_{T/n}.
\end{tikzcd}$$

The functor $S \mapsto M_S$ is required to satisfy the following properties.

- For all truncation sets S, the $\mathbb{W}_S(R)$-module M_S is finitely generated and projective.
- For all truncation sets $T \subset S$:

$$\mathbb{W}_T(R) \otimes_{\mathbb{W}_S(R)} M_S \to M_T$$

  is an isomorphism.
- There is a positive integer a such that there exist morphisms

$$\beta_n : M_{S/n} \to M_S, \tag{4.1.1}$$

  for all positive integers n and all finite truncation sets S, satisfying the following properties:
  - β_n is a morphism of $\mathbb{W}_S(R)$-modules when $M_{S/n}$ is considered as a $\mathbb{W}_S(R)$ module via $F_n : \mathbb{W}_S(R) \to \mathbb{W}_{S/n}(R)$,
  - $\beta_n(\lambda \cdot \phi_n(x)) = n^{a-1} V_n(\lambda) \cdot x$, for all $x \in M_S, \lambda \in \mathbb{W}_{S/n}(R)$,
  - $\phi_n \circ \beta_n = n^a$.

Morphisms between two objects in $\mathcal{C}'_R$ are morphism of functors that are compatible with the $[S \mapsto \mathbb{W}_S(R)]$-module structure and commute with ϕ_n for all positive integers n.

Remark 4.1.2. Note that the β_n are not part of the datum; we can always change $\beta_n \mapsto n^b \beta_n$ for a non-negative integer b.

Proposition 4.1.3. *Let $M \in \mathrm{ob}(\mathcal{C}'_R)$. Let S be a finite truncation set. Fix $a > 0$ and β_n as in 4.1.1.*

(1) *For all positive integers n, m with $(n, m) = 1$ we have*

$$\phi_n \circ \beta_m = \beta_m \circ \phi_n,$$

considered as morphisms $M_{S/m} \to M_{S/n}$.

(2) *For all positive integers n, m we have*

$$\beta_n \circ \beta_m = \beta_{nm},$$

considered as morphisms $M_{S/nm} \to M_S$.

(3) *For all truncation sets $T \subset S$ the following diagram is commutative:*

$$\begin{array}{ccc} M_{S/n} & \xrightarrow{\beta_n} & M_S \\ \downarrow & & \downarrow \\ M_{T/n} & \xrightarrow{\beta_n} & M_T. \end{array}$$

Proof. In any case $\mathbb{W}_S(R)$ is torsion-free and thus M_S is torsion-free for all finite truncation sets S.

For (1). Since $\mathrm{image}(\phi_m) \supset m^a M_{S/m}$ it is sufficient to prove

$$\phi_n \circ \beta_m \circ \phi_m = \beta_m \circ \phi_n \circ \phi_m.$$

This follows from $\beta_m \circ \phi_m = V_m(1) m^{a-1}$ and $\phi_n \circ \phi_m = \phi_m \circ \phi_n$.

For (2). We may argue as in (1) by composing with $\circ\phi_{nm}$.

$$\begin{aligned}\beta_n \circ \beta_m \circ \phi_{nm}(x) &= \beta_n(V_m(1)m^{a-1}\phi_n(x)) \\ &= m^{a-1}n^{a-1}V_{nm}(1)x \\ &= \beta_{nm} \circ \phi_{nm}(x).\end{aligned}$$

For (3). We may argue as in (1) by composing with $\circ\phi_n$. The computation is straightforward. □

Lemma 4.1.4. *Let $f : M \to N$ be a morphism in $\mathcal{C}'_R$, and choose a positive integer a and $\beta_{M,n}$, $\beta_{N,n}$ as in (4.1.1). Then $f_S \circ \beta_{M,n} = \beta_{N,n} \circ f_{S/n}$ for all S, n. In particular, the choice of the β_n in Definition 4.1.1 depends only on the positive integer a.*

Proof. Again, we may use that M_S is torsion-free. Now,

$$\begin{aligned}n^a\beta_n f(x) &= \beta_n(f(n^a x)) = \beta_n(f(\phi_n\beta_n(x))) \\ &= \beta_n\phi_n f(\beta_n(x)) = n^{a-1}V_n(1)f(\beta_n(x)) \\ &= f(n^{a-1}V_n(1)\beta_n(x)) = f(\beta_n\phi_n\beta_n(x)) = n^a f(\beta_n(x)).\end{aligned}$$

□

Proposition 4.1.5 (Tensor products)**.** *For two objects M, N in $\mathcal{C}'_R$ we set*

$$(M \otimes N)_S := M_S \otimes_{\mathbb{W}_S(R)} N_S, \quad \phi_n := \phi_{M,n} \otimes \phi_{N,n}.$$

Then $M \otimes N$ defines an object in $\mathcal{C}'_R$.

Proof. This is a straightforward calculation. We can take $\beta_{M\otimes N,n} = \beta_{M,n} \otimes \beta_{N,n}$, because for all $x \in M_{S/n}$, $y \in N_{S/n}$ and $\lambda \in \mathbb{W}_{S/n}(R)$ we have

$$\begin{aligned}&\beta_{M,n}(\lambda x) \otimes \beta_{N,n}(y) = \beta_{M,n}(x) \otimes \beta_{N,n}(\lambda y) \\ &\Leftrightarrow n\beta_{M,n}(\lambda x) \otimes \beta_{N,n}(y) = n\beta_{M,n}(x) \otimes \beta_{N,n}(\lambda y) \\ &\Leftrightarrow \beta_{M,n}(F_n(V_n(\lambda))x) \otimes \beta_{N,n}(y) = \beta_{M,n}(x) \otimes \beta_{N,n}(F_n(V_n(\lambda))y) \\ &\Leftrightarrow V_n(\lambda)\beta_{M,n}(x) \otimes \beta_{N,n}(y) = \beta_{M,n}(x) \otimes V_n(\lambda)\beta_{N,n}(y).\end{aligned}$$

Moreover, if $\phi_{M,n} \circ \beta_{M,n} = n^a$ and $\phi_{N,n} \circ \beta_{N,n} = n^b$ then

$$\begin{aligned}(\phi_{M,n} \otimes \phi_{N,n}) \circ (\beta_{M,n} \otimes \beta_{N,n}) &= n^{a+b}, \\ (\beta_{M,n} \otimes \beta_{N,n}) \circ (\lambda\phi_{M,n} \otimes \phi_{N,n}) &= V_n(\lambda)V_n(1)n^{a+b-2} = V_n(\lambda)n^{a+b-1}.\end{aligned}$$

□

The tensor product equips $\mathcal{C}'_R$ with the structure of a $\otimes$-category with identity object $\mathbf{1}$, where

$$\mathbf{1}_S := \mathbb{W}_S(R), \qquad \phi_{\mathbf{1},n} = F_n.$$

Definition 4.1.6. (Tate objects) Let b be a non-negative integer. We define the object $\mathbf{1}(-b)$ in $\mathcal{C}'_R$ by

$$\mathbf{1}(-b)_S := \mathbb{W}_S(R), \qquad \phi_{\mathbf{1}(-b),n} = n^b F_n.$$

For an object M in $\mathcal{C}'_R$, the modules M_S are torsion-free, hence

$$\operatorname{Hom}_{\mathcal{C}'_R}(M, N) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{C}'_R}(M \otimes \mathbf{1}(-b), N \otimes \mathbf{1}(-b))$$

is an isomorphism.

Definition 4.1.7. We denote by $\mathcal{C}_R$ the category with objects $M(b)$, where M is an object in $\mathcal{C}'_R$ and $b \in \mathbb{Z}$. As morphisms we set

$$\mathrm{Hom}_{\mathcal{C}_R}(M(b_1), N(b_2)) = \mathrm{Hom}_{\mathcal{C}'_R}(M \otimes \mathbf{1}(b_1 - c), N \otimes \mathbf{1}(b_2 - c)),$$

where $c \in \mathbb{Z}$ is such that $b_1 - c, b_2 - c \leq 0$.

The category $\mathcal{C}_R$ is additive and via $M \mapsto M(0)$ the category $\mathcal{C}'_R$ is a full subcategory of $\mathcal{C}_R$. For $M \in \mathcal{C}'_R$, we have $M(-b) = M \otimes \mathbf{1}(-b)$ if b is non-negative. For an integer b, the functor

$$\mathcal{C}_R \to \mathcal{C}_R, \quad M(n) \mapsto M(n+b)$$

is an equivalence and has $M(n) \mapsto M(n-b)$ as inverse functor.

For $M(b_1), N(b_2)$ in $\mathcal{C}_R$ we set

$$M(b_1) \otimes N(b_2) := (M \otimes N)(b_1 + b_2).$$

The tensor product equips $\mathcal{C}_R$ with the structure of a $\otimes$-category with identity object $\mathbf{1}$.

4.1.8. *Internal Hom.* The reason for introducing the new category $\mathcal{C}_R$ is the internal Hom construction.

Let M, N be two objects in $\mathcal{C}'_R$, fix positive integers a_M, a_N and $\beta_{n,M}, \beta_{n,N}$ as in (4.1.1). In a first step we are going to define an object $\underline{\mathrm{Hom}}'(M,N)$ in $\mathcal{C}'_R$ that depends on the choice of a_M. We set

$$\underline{\mathrm{Hom}}'(M,N)_S := \mathrm{Hom}_{\mathbb{W}_S(R)}(M_S, N_S).$$

We note that

$$\mathrm{Hom}_{\mathbb{W}_S(R)}(M_S, N_S) \otimes_{\mathbb{W}_S(R)} \mathbb{W}_T(R) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{W}_T(R)}(M_T, N_T),$$

since M_S is locally free. We define

$$\begin{gathered}\phi_n : \mathrm{Hom}_{\mathbb{W}_S(R)}(M_S, N_S) \to \mathrm{Hom}_{\mathbb{W}_{S/n}(R)}(M_{S/n}, N_{S/n}) \\ \phi_n(f) := \phi_n \circ f \circ \beta_n.\end{gathered}$$

This definition depends on a_M. It is easy to check that $\underline{\mathrm{Hom}}'(M,N)$ is an object in $\mathcal{C}'_R$ (take $\beta_n(f) := \beta_n \circ f \circ \phi_n$ and $a = a_M + a_N$). We set

$$\underline{\mathrm{Hom}}(M,N) := \underline{\mathrm{Hom}}'(M,N)(a_M) \tag{4.1.2}$$

as an object in $\mathcal{C}_R$. In view of Lemma 4.1.4 this definition is independent of any choices. For two objects $M(b_1), N(b_2)$ in $\mathcal{C}_R$ we set

$$\underline{\mathrm{Hom}}(M(b_1), N(b_2)) := \underline{\mathrm{Hom}}(M,N)(b_2 - b_1).$$

4.1.9. For three objects M, N, P in $\mathcal{C}_R$ we have an obvious natural isomorphism

$$\underline{\mathrm{Hom}}(M \otimes N, P) = \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(N,P)).$$

Proposition 4.1.10. *For objects M, N in $\mathcal{C}_R$ we have a natural isomorphism*

$$\mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(M,N)) \to \mathrm{Hom}(M,N).$$

Proof. We may assume that $M, N \in \mathcal{C}'_R$. Fix a_M and $\beta_{M,n}$ as in (4.1.1). We need to show that

$$\mathrm{Hom}(\mathbf{1}(-a_M), \underline{\mathrm{Hom}}'(M,N)) = \mathrm{Hom}(M,N),$$

and know that

$$\begin{aligned}\mathrm{Hom}(\mathbf{1}(-a_M), \underline{\mathrm{Hom}}'(M,N)) = \{[S \mapsto f_S] \mid f_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_T(R) = f_T \text{ for all } T \subset S,\\ \phi_{N,n} \circ f_S \circ \beta_{M,n} = n^{a_M} f_{S/n} \quad \text{for all } S, n.\}\end{aligned}$$

Since $\phi_{M,n}(M_S) \supset n^{a_M} M_{S/n}$ we have

$$\begin{aligned}\phi_{N,n} \circ f_S \circ \beta_{M,n} = n^{a_M} f_{S/n} &\Leftrightarrow \phi_{N,n} \circ f_S \circ \beta_{M,n} \circ \phi_{M,n} = n^{a_M} f_{S/n} \circ \phi_{M,n}\\ &\Leftrightarrow \phi_{N,n} \circ f_S \circ n^{a_M - 1} V_n(1) = n^{a_M} f_{S/n} \circ \phi_{M,n}\\ &\Leftrightarrow n^{a_M} \phi_{N,n} \circ f_S = n^{a_M} f_{S/n} \circ \phi_{M,n}\\ &\Leftrightarrow \phi_{N,n} \circ f_S = f_{S/n} \circ \phi_{M,n}.\end{aligned}$$

□

For $M \in \mathcal{C}_R$ we define the dual by

$$M^\vee := \underline{\mathrm{Hom}}(M, \mathbf{1}).$$

It equips $\mathcal{C}_R$ with the structure of rigid $\otimes$-category. We have

$$M^\vee \otimes N = \underline{\mathrm{Hom}}(M,N).$$

4.1.11. Our motivation for introducing $\mathcal{C}_R$ comes from geometry.

Proposition 4.1.12. *Assumptions as in Theorem 3.3.1. For all $i \geq 0$ the assignment*

$$S \mapsto H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}}), \quad n \mapsto \phi_n,$$

defines an object in $\mathcal{C}'_R$.

Proof. Theorem 3.3.1 implies that these modules are projective and finitely generated. For the construction of ϕ_n and β_n see Section 3.3.8. Together with Proposition 3.3.7 this proves the claim. □

Definition 4.1.13. Let $X \to \mathrm{Spec}(R)$ be a morphism such that the assumptions of Theorem 3.3.1 are satisfied. For all i, we denote by $H^i_{dRW}(X/R)$ the object in $\mathcal{C}_R$ that is given by $S \mapsto H^i(X, \mathbb{W}_S\Omega^*_{X/\mathbb{Z}})$ (Proposition 4.1.12). We call $H^*_{dRW}(X/R)$ the *(big) de Rham-Witt cohomology* of X.

Lemma 4.1.14. *Let $X \to \mathrm{Spec}(R)$ be a morphism such that the assumptions of Theorem 3.3.1 are satisfied. Set $d := \dim X - 1$. Then*

$$H^i_{dRW}(X/R) = 0 \qquad \textit{for all } i > 2d.$$

Proof. We know that $\mathbb{W}_S\Omega^i_{X/\mathbb{Z}} = 0$ if $i \geq \dim X = d+1$. Indeed, we can use Lemma 2.2.6 and the explicit version of the big de Rham-Witt complex for the affine space (Theorem 2.2.12) in order to conclude. The vanishing $H^{d+1}(X, \mathbb{W}_S\Omega^j_{X/\mathbb{Z}}) = 0$ is part of Theorem 3.2.2. □

4.1.15. Let X, Y be smooth projective schemes over R such that the assumptions of Theorem 3.3.1 are satisfied for X and Y. The multiplication

$$R\Gamma(\mathbb{W}_S\Omega^*_{X/\mathbb{Z}}) \times R\Gamma(\mathbb{W}_S\Omega^*_{Y/\mathbb{Z}}) \to R\Gamma(\mathbb{W}_S\Omega^*_{X\times_R Y/\mathbb{Z}})$$

induces a morphism in $\mathcal{C}_R$:

$$H^i_{dRW}(X/R) \otimes H^j_{dRW}(Y/R) \to H^{i+j}_{dRW}(X \times_R Y/R). \tag{4.1.3}$$

4.2. **The tangent space functor.** We have a functor of rigid $\otimes$-categories

$$T : \mathcal{C}_R \to \text{(projective finitely generated } R\text{-modules)}$$
$$T(M(n)) := M_{\{1\}}.$$

Proposition 4.2.1. *The functor T is conservative.*

Proof. It is sufficient to consider a morphism $f : M \to N$ in $\mathcal{C}'_R$. We need to show that $f_S : M_S \to N_S$ is an isomorphism provided that $f_{\{1\}}$ is an isomorphism. We may choose a positive integer a and $\beta_{M,n}$, $\beta_{N,n}$ as in (4.1.1). By Lemma 4.1.4 the morphism f commutes with β_n.

Let $n := \max\{s \mid s \in S\}$; by induction we know that f_T is an isomorphism for $T = S\backslash\{n\}$. Since R is étale over $\mathbb{Z}$ or over $\mathbb{Z}_{(p)}$, we know that $\ker(\mathbb{W}_S(R) \to \mathbb{W}_T(R)) = (V_n(1))$ and thus $M_T = M_S/V_n(1)M_S$ and $N_T = N_S/V_n(1)N_S$. It suffices to show that

$$V_n(1)M_S \xrightarrow{f_S} V_n(1)N_S \tag{4.2.1}$$

is an isomorphism. If $f_S(V_n(1)x) = 0$ then $n^{a-1}V_n(1)f_S(x) = 0$ and therefore $\beta_n(f_{\{1\}}(\phi_n(x))) = \beta_n\phi_n f_S(x)$ vanishes. Since β_n is injective, we conclude $\phi_n(x) = 0$, hence

$$0 = \beta_n\phi_n(x) = n^{a-1}V_n(1)x,$$

which implies $V_n(1)x = 0$.

For the surjectivity of (4.2.1) we note that, by induction, for every $y \in N_S$ there is $x \in M_S$ with $f_S(x) - y \in V_n(1)N_S$. Therefore it suffices to show that $V_n(1)^a N_S$ is contained in the image of f_S. Now, $V_n(1)^a = n^{a-1}V_n(1) = \beta_n\phi_n$ and thus

$$V_n(1)^a y = f_S(\beta_n f^{-1}_{\{1\}}\phi_n(y)).$$

□

Corollary 4.2.2. *Let X, Y be smooth proper schemes over R such that the assumptions of Theorem 3.3.1 are satisfied for X and Y. If*

$$\bigoplus_{i+j=n} H^i_{dR}(X/R) \otimes_R H^j_{dR}(X/R) \to H^n_{dR}(X \times_R Y/R)$$

is an isomorphism then

$$\bigoplus_{i+j=n} H^i_{dRW}(X/R) \otimes H^j_{dRW}(X/R) \to H^n_{dRW}(X \times_R Y/R)$$

(see (4.1.3)) is an isomorphism in $\mathcal{C}_R$.

Proof. This is an application of Proposition 4.2.1, because

$$T(H^i_{dRW}(-/R)) = H^i_{dR}(-/R).$$

□

Proposition 4.2.3. *The functor T is faithful.*

Proof. It is sufficient to consider a morphism $f : M \to N$ in $\mathcal{C}'_R$. We need to show that $f_S : M_S \to N_S$ vanishes provided that $f_{\{1\}}$ is zero. We may choose a positive integer a and $\beta_{M,n}$, $\beta_{N,n}$ as in (4.1.1). By Lemma 4.1.4 the morphism f commutes with β_n.

Let $n := \max\{s \in S\}$; by induction we know that $f_T = 0$ for $T = S\backslash\{n\}$. We know that $\ker(N_S \to N_T) = V_n(1)N_S$, so that $f_S(x) = V_n(1)y$ for all $x \in M_S$. Since

$$0 = f_{\{1\}} \circ \phi_n(x) = \phi_n \circ f_S(x) = n \cdot \phi_n(y),$$

we conclude $\phi_n(y) = 0$ and $n^{a-1}V_n(1)y = 0$, hence $f_S(x) = 0$. □

4.2.4. Our next goal is to show that cokernels exist for morphisms f in $\mathcal{C}_R$ such that $\mathrm{coker}(T(f))$ is projective (that is, torsion-free).

The following proposition shows that a $\mathcal{C}'_R$-module, where R is an étale $\mathbb{Z}_{(p)}$-algebra, is determined by the p-typical part, that is, on its values for truncation sets consisting of p-powers.

Proposition 4.2.5. *Let R be étale over $\mathbb{Z}_{(p)}$. Let M, N be $\mathcal{C}'_R$-modules.*

(1) *Via the equivalence of Corollary 1.0.22:*

$$M_S \mapsto \bigoplus_{n\geq 1,(n,p)=1} M_{(S/n)_p}.$$

(2) *If $f : M \to N$ is a morphism in $\mathcal{C}'_R$ then $f_S \mapsto \bigoplus_{n\geq 1,(n,p)=1} f_{(S/n)_p}$ via the equivalence Corollary 1.0.22.*

Proof. The proof of (1) is as in Proposition 1.0.21. First one proves that the projection $\epsilon_1 M_S \to M_{S_p}$ is an isomorphism (see Notation 1.0.20 for S_p). The second step is the isomorphism

$$\phi_n : \epsilon_n M_S \to \epsilon_1 M_{S/n},$$

with $\frac{\epsilon_n}{n^a}\beta_n$ as inverse.

Statement (2) is obvious. □

Proposition 4.2.6. *Let R be an étale $\mathbb{Q}$-algebra. Let M, N be $\mathcal{C}'_R$-modules.*

(1) *Via the equivalence of Corollary 1.0.24:*

$$M_S \mapsto \bigoplus_{n\in S} M_{\{1\}}.$$

(2) *If $f : M \to N$ is a morphism in $\mathcal{C}'_R$ then $f_S \mapsto \bigoplus_{n\in S} f_{\{1\}}$ via the equivalence of Corollary 1.0.24.*

Proof. Straightforward. □

Proposition 4.2.7 (Localization)**.** *Let R be an étale $\mathbb{Z}$-algebra. Let $\mathfrak{p} \subset R$ be a maximal ideal. The assignment*

$$[S \mapsto M_S] \mapsto [S \mapsto M_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_{\mathfrak{p}})]$$
$$[S \mapsto f_S] \mapsto [S \mapsto f_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_{\mathfrak{p}})].$$

defines a functor $\mathcal{C}'_R \to \mathcal{C}'_{R_{\mathfrak{p}}}$.

Proof. This is an easy calculation. We set $\phi_n = \phi_{M,n} \otimes F_n$ and $\beta_n = \beta_{M,n} \otimes V_n$. □

In the obvious way it extends to a functor of rigid $\otimes$-categories

$$\mathcal{C}_R \to \mathcal{C}_{R_\mathfrak{p}}.$$

4.2.8. *Existence of cokernels and kernels.*

Proposition 4.2.9. *Let R be an étale $\mathbb{Z}$-algebra. Let $f : M \to N$ be a morphism in $\mathcal{C}_R$ such that* $\operatorname{coker}(T(f))$ *is a projective R-module. Then* $\operatorname{coker}(f)$ *exists in $\mathcal{C}_R$ and* $T(\operatorname{coker}(f)) = \operatorname{coker}(T(f))$.

Proof. Without loss of generality we may assume that $f \in \mathcal{C}'_R$. We may choose a positive integer a and $\beta_{M,n}$, $\beta_{N,n}$ as in (4.1.1). By Lemma 4.1.4 the morphism f commutes with β_n.

We set $C_S := \operatorname{coker}(f_S : M_S \to N_S)$ and we need to show that $S \mapsto C_S$ defines an object in $\mathcal{C}'_R$. The $\phi_{C,n}$ and $\beta_{C,n}$ are induced and satisfy the equalities $\beta_n \circ \phi_n = V_n(1)n^{a-1}$ and $\phi_n \circ \beta_n = n^a$.

As the next step we will show that

$$\mathbb{W}_T(R) \otimes_{\mathbb{W}_S(R)} C_S \to C_T \tag{4.2.2}$$

is an isomorphism for two finite truncation sets $T \subset S$. We can restrict ourselves to the case $T = S\backslash\{n\}$ for one element $n \in S$. Since R is étale over $\mathbb{Z}$ we have $V_n(1)\mathbb{W}_S(R) = \ker(\mathbb{W}_S(R) \to \mathbb{W}_T(R))$ and we need to show

$$C_S/V_n(1)C_S \xrightarrow{\cong} C_T.$$

Surjectivity is obvious. For the injectivity: let $y \in N_S$ be such that the projection y_T in N_T is contained in $f_T(M_T)$, say $y_T = f(x_T)$. Choose a lifting $x \in M_S$ of x_T in order to obtain $y - f(x) \in \ker(N_S \to N_T) = V_n(1)N_S$.

The only property that is not obvious is that the $\mathbb{W}_S(R)$-module C_S is projective. Since C_S is finitely generated and $\mathbb{W}_S(R)$ is noetherian, it suffices to prove this after localizing at every maximal ideal $\mathfrak{m}$ of $\mathbb{W}_S(R)$. In view of Lemma 1.0.26 we may replace R by $R_\mathfrak{p}$ for a maximal ideal $\mathfrak{p}$ in R. That is, we consider

$$S \mapsto C_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p}) = \operatorname{coker}(M_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p}) \to N_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p})).$$

In view of Proposition 4.2.7 we know that $S \mapsto M_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p})$ and $S \mapsto N_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p})$ are objects in $\mathcal{C}'_{R_\mathfrak{p}}$.

Set $(p) = \mathfrak{p} \cap \mathbb{Z}$, by using Proposition 4.2.5, we can reduce to the case where S consists of p-powers. In this case $\mathbb{W}_S(R_\mathfrak{p})$ is a local ring (Lemma 1.0.27). We claim that there is a $\mathbb{W}_S(R_\mathfrak{p})$-basis $v_1, \dots, v_s$ of $M_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p})$ and a $\mathbb{W}_S(R_\mathfrak{p})$-basis $w_1, \dots, w_t$ for $N_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p})$ such that

$$f_S(v_i) = \begin{cases} w_i & \text{if } i \le r, \\ 0 & \text{if } i > r, \end{cases}$$

where $r = \operatorname{rank}(f_{\{1\}})$. Let us prove the claim by induction. The case $S = \{1\}$ holds by assumption.

For the case $S = \{1, p, p^2, \dots, p^n\}$ we know the existence of such two basis for S/p. Any liftings $\tilde{v}_1, \dots, \tilde{v}_s$ resp. $\tilde{w}_1, \dots, \tilde{w}_t$ will form a basis for $M_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p})$ resp. $N_S \otimes_{\mathbb{W}_S(R)} \mathbb{W}_S(R_\mathfrak{p})$. Obviously, we can take $\tilde{w}_i := f_S(\tilde{v}_i)$ for all $i \le r$, and choose $\tilde{v}_j$, for $j > r$, such that

$$f_S(\tilde{v}_j) = V_{p^n}(1) \sum_{k>r} B_{jk} \tilde{w}_k. \tag{4.2.3}$$

We want to show that the coefficients $V_{p^n}(1)B_{jk}$ vanish. Let us denote by $\bar{v}_i$ the image of $\tilde{v}_i$ in $M_{\{1\}} \otimes_R R_{\mathfrak{p}}$. We define the matrix $\Lambda = (\lambda_{ij}) \in \mathrm{Mat}(s \times s, R_{\mathfrak{p}})$ by

$$p^{na}\bar{v}_i = \sum_{j=1}^{s} \lambda_{ij}\phi_{p^n}(\tilde{v}_j). \tag{4.2.4}$$

Considered as element in $\mathrm{Mat}(s \times s, \mathrm{Quot}(R_{\mathfrak{p}}))$ the matrix Λ is invertible. After applying $f_{\{1\}}$ to (4.2.4) we obtain for all $i > r$:

$$0 = \sum_{j \le r} \lambda_{ij}\phi_{p^n}(\tilde{w}_j) + p^n \sum_{k>r}\sum_{j>r} \lambda_{ij}F_{p^n}(B_{jk})\phi_{p^n}(\tilde{w}_k).$$

The elements $(\phi_{p^n}(\tilde{w}_j))_j$ form a $\mathrm{Quot}(R_{\mathfrak{p}})$-basis of $N_{\{1\}} \otimes_R \mathrm{Quot}(R_{\mathfrak{p}})$, hence

$$\lambda_{ij} = 0 \qquad \text{for } i > r \text{ and } j \le r, \tag{4.2.5}$$

$$\sum_{j>r} \lambda_{ij}F_{p^n}(B_{jk}) = 0 \qquad \text{for } i > r \text{ and } k > r. \tag{4.2.6}$$

Condition (4.2.5) implies that

$$\Lambda = \begin{pmatrix} \Lambda_1 & ? \\ 0 & \Lambda_2 \end{pmatrix},$$

where $\Lambda_1 \in \mathrm{Mat}(r \times r, R_{\mathfrak{p}})$ and $\Lambda_2 \in \mathrm{Mat}((s-r) \times (s-r), R_{\mathfrak{p}})$, hence $\Lambda_2 \in \mathrm{GL}_{s-r}(\mathrm{Quot}(R_{\mathfrak{p}}))$. Therefore (4.2.6) implies $F_{p^n}(B_{jk}) = 0$ for all $j, k > r$, and thus $V_{p^n}(1)B_{jk} = 0$. In view of (4.2.3) we are done. □

Corollary 4.2.10. *Let R be an étale $\mathbb{Z}$-algebra. Let $f : M \to N$ be a morphism in $\mathcal{C}_R$ such that* $\mathrm{coker}(T(f))$ *is a projective R-module. Then* $\ker(f)$ *exists in $\mathcal{C}_R$ and* $T(\ker(f)) = \ker(T(f))$.

Proof. Follows from Proposition 4.2.9 by taking duals. □

4.2.11. For a morphism $R \to R'$ between étale $\mathbb{Z}$-algebras we get an obvious functor of rigid $\otimes$-categories:

$$\mathcal{C}_R \to \mathcal{C}_{R'}. \tag{4.2.7}$$

Definition 4.2.12. Let K be a number field and A its ring of integers. We define the category $\mathcal{E}_K$ with objects $M_U \in \mathrm{ob}(\mathcal{C}_{\Gamma(U,\mathcal{O}_{\mathrm{Spec}(A)})})$ where $U \subset \mathrm{Spec}(A)$ is a non-empty open that is étale over $\mathrm{Spec}(\mathbb{Z})$. The morphisms are defined by

$$\mathrm{Hom}_{\mathcal{E}_K}(M_U, N_{U'}) := \varinjlim_{\emptyset \ne V \subset U \cap U' \text{ open}} \mathrm{Hom}_{\mathcal{C}_{\Gamma(V,\mathcal{O}_{\mathrm{Spec}(A)})}}(M_U\,|_V, N_{U'}\,|_V),$$

where $|_V$ denotes the restriction functor (4.2.7).

By definition, $\mathcal{E}_K$ is a rigid $\otimes$-category.

Proposition 4.2.13. *Let K be a number field. The category $\mathcal{E}_K$ is an abelian rigid $\otimes$-category with* $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$. *The functor*

$$T_K : \mathcal{E}_K \to \textit{(finite dimensional } K\textit{-vector spaces)}$$
$$T_K(M_U) = T(M_U) \otimes_{\Gamma(U,\mathcal{O}_{\mathrm{Spec}(A)})} K$$

is a fibre functor. In particular, the category $\mathcal{E}_{\mathbb{Q}}$ is a neutral tannakian category.

Proof. That $\mathcal{E}_K$ is an abelian category follows from Proposition 4.2.9, Corollary 4.2.10, and Proposition 4.2.1. In view of Proposition 4.2.3, the functor T_K is an exact faithful $\otimes$-functor.

We still have to prove that $\mathrm{End}_{\mathcal{E}_K}(\mathbf{1}) = \mathbb{Q}$. Recall that A denotes the ring of integers of K; it is sufficient to show that

$$\mathrm{End}_{\mathcal{C}_{A[N^{-1}]}}(\mathbf{1}) = \mathbb{Z}[N^{-1}], \tag{4.2.8}$$

for all N such that $A[N^{-1}]$ is étale over $\mathbb{Z}[N^{-1}]$. This is equivalent to

$$B := \{b \in \mathbb{W}(A[N^{-1}]) \mid F_p(b) = b \text{ for all primes } p\} = \mathbb{Z}[N^{-1}],$$

where $\mathbb{W} = \varprojlim_S \mathbb{W}_S$. Obviously, B is a subring. Since T is faithful, the map

$$B \to \mathbb{W}(A[N^{-1}]) \to \mathbb{W}_{\{1\}}(A[N^{-1}]) = A[N^{-1}]$$

is injective, hence B is integral and finite over $\mathbb{Z}[N^{-1}]$. We claim that

$$b^p - b \in pB$$

for all $b \in B$ and all primes p. Indeed, $\mathbb{W}(A[N^{-1}])$ is a λ-ring and in particular F_p is a lifting of the absolute Frobenius: $F_p(b) \in b^p + p\mathbb{W}(A[N^{-1}])$. Because $\mathbb{W}(A[N^{-1}])$ is p-torsion-free we get $p\mathbb{W}(A[N^{-1}]) \cap B = pB$, and the claim is proved.

Set $r = \mathrm{rank}_{\mathbb{Z}[N^{-1}]}(B)$; for all primes p such that $B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is étale over $\mathbb{Z}_{(p)}$ we have $B/pB = \prod_{i=1}^r \mathbb{F}_p$, which contradicts Chebotarev's density theorem unless $r = 1$. In the case $r = 1$ we obtain $B = \mathbb{Z}[N^{-1}]$. □

4.2.14. Let K be a number field. For all $i \geq 0$ we define a functor

$$(\text{smooth projective varieties over } K)^{\mathrm{op}} \to \mathcal{E}_K, \quad X \mapsto H^i_{dRW}(X/K),$$

as follows. Let R be a finite and étale $\mathbb{Z}[N^{-1}]$-algebra, for some $N \neq 0$, such that

- $R \otimes \mathbb{Q} = K$,
- $X \to \mathrm{Spec}(K)$ admits a model $f : Y \to \mathrm{Spec}(R)$ with the following properties:
  - f is smooth and projective,
  - the de Rham cohomology $H^*_{dR}(X/R)$ is a projective R-module.

Such an R can always be found. In view of Proposition 4.1.12 and Definition 4.1.13 we obtain an object $H^i_{dRW}(Y/R)$ in $\mathcal{C}_R$. We denote the image in $\mathcal{E}_K$ by $H^i_{dRW}(X/K)$. The independence of the choice of the model $f : Y \to \mathrm{Spec}(R)$ is proved by comparing with another model $f' : Y' \to \mathrm{Spec}(R')$ via a canonical isomorphism over a suitable localization R''. In the same way we can prove the functoriality.

4.3. **Poincaré duality.**

Proposition 4.3.1. *Let R be an étale $\mathbb{Z}$-algebra. Let X be a smooth projective scheme over R such that $H^*_{dR}(X/R)$ is a projective R-module. Suppose that X is connected, and set $d = \dim X - 1$. There is an isomorphism*

$$H^{2d}_{dRW}(X/R) \cong H^0_{dRW}(X/R) \otimes \mathbf{1}(-d)$$

and a natural morphism

$$H^{2d}_{dRW}(X/R) \to \mathbf{1}(-d)$$

in $\mathcal{C}_R$.

Proof. 1. Step: Reduction to X/R geometrically integral. Set $L = H^0(X, \mathcal{O}_X)$, L is a finite étale R-algebra. It suffices to show the existence of an isomorphism

$$H^{2d}_{dRW}(X/L) \xrightarrow{\phi} \mathbf{1}(-d) \tag{4.3.1}$$

in $\mathcal{C}_L$ such that $\phi_{\{1\}}$ is the trace map. From (4.3.1) we obtain in $\mathcal{C}_R$:

$$H^{2d}_{dRW}(X/R) \xrightarrow{\cong} H^0_{dRW}(X/R) \otimes \mathbf{1}(-d) \xrightarrow{tr \otimes id} \mathbf{1}(-d), \tag{4.3.2}$$

with $tr : H^0_{dRW}(X/R) \to \mathbf{1}$ being defined by the usual trace map $H^0_{dRW}(X/R)_S = \mathbb{W}_S(L) \to \mathbb{W}_S(R)$. The morphism (4.3.2) is functorial because it induces the usual trace map after evaluation at $\{1\}$. Therefore we may assume $R = L$ in the following. In particular, R is integral.

2. Step: Reduction to a local statement. Set $M := H^{2d}_{dRW}(X/R)$. Let $M \otimes \mathbb{Q} \in \mathcal{C}'_{\mathbb{Q}}$ denote the object $[S \mapsto M_S \otimes_{\mathbb{Z}} \mathbb{Q}]$. In view of Proposition 4.2.6 there is a unique morphism

$$\phi : \mathbf{1}(-d) \to M \otimes \mathbb{Q}$$

such that $\phi_{\{1\}}(1) = \mathrm{Tr}^{-1}(1) \in H^{2d}_{dR}(X/R)$. Since ϕ corresponds to a system $[S \mapsto \phi_S(1) =: \tau_S]$, with $\tau_S \in M_S \otimes \mathbb{Q}$, it is sufficient to show that $\tau_S \in M_S \otimes \mathbb{Z}_{(p)}$ for all finite truncation sets and all primes p. If p is a unit of R then this is trivial, so that we may assume $pR \neq R$. The uniqueness of ϕ implies that it suffices to show the existence of a morphism

$$\phi_p : \mathbf{1}(-d) \to M \otimes \mathbb{Z}_{(p)}$$

in $\mathcal{C}'_{R \otimes \mathbb{Z}_{(p)}}$ such that $\phi_{p,\{1\}}(1) = \mathrm{Tr}^{-1}(1)$. Moreover, we may define ϕ_p only on the p-typical part (Proposition 4.2.5). And we may pass from R to a localization R' such that $R' \otimes \mathbb{Z}_{(p)} = R \otimes \mathbb{Z}_{(p)}$.

3. Step: Construction of a multiple of the desired morphism. We fix a prime p with $pR \neq R$. After passing to a localization R' of R we may suppose that there exists a generically finite dominant morphism $f : X \to \mathbb{P}^d_R$. By using the explicit version of the de Rham-Witt complex for λ-rings (Theorem 2.2.12) it is not difficult to prove the existence of

$$\mathbf{1}(-d) \xrightarrow{\cong} H^{2d}_{dRW}(\mathbb{P}^d_R/R)$$

such that $1 \mapsto \mathrm{Tr}^{-1}(1)$ after evaluation at $\{1\}$. We have $f^*(\mathrm{Tr}^{-1}(1)) = \deg(f)\mathrm{Tr}^{-1}(1)$, hence $\deg(f)\tau_S \in M_S \otimes \mathbb{Z}_p$ for all S.

4. Step: Final step. We set $M_p := \varprojlim_n (M_{\{1,p,\ldots,p^{n-1}\}} \otimes \mathbb{Z}_{(p)})$, it is a free $W(R \otimes \mathbb{Z}_{(p)})$-module of rank 1, and we can find a basis e such that $\pi_{\{1\}}(e) = \mathrm{Tr}^{-1}(1)$. We denote by $\pi_{\{1\}} : M_p \to H^{2d}_{dR}(X/R) \otimes \mathbb{Z}_{(p)}$ the evident map.

It follows from the third step of the proof that we have the element $\tau'_p := (\deg(f)\tau_S)_S \in M_p$ at our disposal. It satisfies $\phi_p(\tau'_p) = p^d \tau'_p$.

Set $\hat{R} := \varprojlim_n R/p^n$. Lemma 3.3.10 shows

$$M_p \otimes_{W(R \otimes \mathbb{Z}_{(p)})} W(\hat{R}) \cong H^{2d}_{crys}(X \otimes_R (R/p)/W(R/p)) \otimes_{W(R/p)} W(\hat{R}).$$

We know that $H^{2d}_{crys}(X \otimes_R (R/p)/W(R/p)) = W(R/p)e'$ with $\phi_p(e') = p^d e'$, and we get $M_p \otimes_{W(R \otimes \mathbb{Z}_{(p)})} W(\hat{R}) = W(\hat{R})e'$.

Write $\tau'_p = \delta \cdot e'$ with $\delta \in W(\hat{R})$. Since $F_p(\delta) = \delta$, Lemma 4.3.2 shows $\delta = \rho(\delta')$ for $\delta' \in W((R/p)^{\mathrm{Frob}-1})$. Let us extend $\pi_{\{1\}}$ to

$$\pi_{\{1\}} : M_p \otimes_{W(R \otimes \mathbb{Z}_{(p)})} W(\hat{R}) \to H^{2d}_{dR}(X/R) \otimes_R \hat{R}.$$

We obtain

$$\pi_{\{1\}}(\tau'_p) = \pi_{\{1\}}(\delta \cdot e') = \delta' \cdot \pi_{\{1\}}(e').$$

Since $\pi_{\{1\}}(e')$ is a generator of $H^{2d}_{dR}(X/R) \otimes_R \hat{R}$, it follows that $\delta' = \deg(f)\delta''$ with $\delta'' \in W((R/p)^{\mathrm{Frob}-1})$, hence $\delta = \deg(f) \cdot \rho(\delta'')$.

Finally, write $\tau'_p = \lambda \cdot e$ with $\lambda \in W(R \otimes \mathbb{Z}_{(p)})$, and $e' = a \cdot e$ with $a \in W(\hat{R})$. Obviously, $\lambda = \deg(f)\rho(\delta'')a$. This implies $\rho(\delta'')a \in W(R \otimes \mathbb{Z}_{(p)})$. Indeed, it's sufficient to show that the image of $\rho(\delta'')a$ in $W_n(\hat{R})$ is contained in $W_n(R \otimes \mathbb{Z}_{(p)})$ for all n. And this follows from the fact that $W_n(R \otimes \mathbb{Z}_{(p)}) \to W_n(\hat{R})$ is faithfully flat (Lemma 3.3.2, Lemma 1.0.27). The element $\rho(\delta'')a \cdot e \in M_p$ solves our problem. ☐

Lemma 4.3.2. *Let R be a étale $\mathbb{Z}_{(p)}$-algebra, set $\hat{R} := \varprojlim_n R/p^n$. The map*

$$W((R/p)^{\mathrm{Frob}-1}) \to W(\hat{R})^{F_p - 1}$$

is bijective.

Proof. We use the notation

$$(R/p)^{\mathrm{Frob}-1} = \{x \in R/p \mid \mathrm{Frob}(x) = x\},$$
$$W(\hat{R})^{F_p-1} = \{x \in W(\hat{R}) \mid F_p(x) = x\}.$$

We have $W(R/p) = \hat{R}$. The ring homomorphism $\rho : \hat{R} \to W(\hat{R})$ is induced by the lifting of the Frobenius $\sigma : \hat{R} \to \hat{R}$, and satisfies $F_p \circ \rho = \rho \circ \sigma$. Therefore the map in the assertion is well-defined, we denote it by η.

It is easy to see that for all $i \geq 0$:

$$gh_{p^i} \circ \rho(x) = \lim_{n \to \infty} gh_{p^i} \circ F_p^n \left(\widetilde{\sigma^{-n}(x)} \right),$$

where $\widetilde{\sigma^{-n}(x)}$ is a lifting of $\sigma^{-n}(x)$ via the map $W(\hat{R}) \to W(R/p)$. The sequence $\left(gh_{p^i} \circ F_p^n \left(\widetilde{\sigma^{-n}(x)} \right) \right)_n$ converges p-adically.

Since the ghost map gh is injective, it is evident that

$$\psi : W(\hat{R})^{F_p - 1} \to W((R/p)^{\mathrm{Frob}-1}),$$

induced by $\hat{R} \to R/p$, defines an inverse for η. ☐

Corollary 4.3.3. *Let R be an étale $\mathbb{Z}$-algebra. Let $X \to \mathrm{Spec}(R)$ be a smooth projective morphism such that $H^*_{dR}(X/R)$ is a projective R-module. Suppose that X is connected, set $d := \dim X - 1$. If the canonical map*

$$H^i_{dR}(X/R) \to \mathrm{Hom}_R(H^{2d-i}_{dR}(X/R), R) \tag{4.3.3}$$

is an isomorphism, then the same holds for the de Rham-Witt cohomology:

$$H^i_{dRW}(X/R) \xrightarrow{\cong} \underline{\mathrm{Hom}}(H^{2d-i}_{dRW}(X/R), \mathbf{1}(-d)). \tag{4.3.4}$$

Proof. In view of Proposition 4.3.1 and (4.1.3) we get a morphism in $\mathcal{C}_R$:

$$H^i_{dRW}(X/R) \otimes H^{2d-i}_{dRW}(X/R) \to H^{2d}_{dRW}(X/R) \to \mathbf{1}(-d)$$

inducing (4.3.4). Now, $T(4.3.4) = (4.3.3)$ proves the claim. ☐

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